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Foundations of Control

PERHAPS instigated by the concept of adaptive control, many recent papers have been devoted to the philosophy of automatic control. For example, in the Dallas Control Conference there were papers entitled "Fundamental Theory of Automatic Linear Feedback Control Systems," "Impact of Information Conversion on Control," "General Approach to Control Theory Based on the Methods of Lyapunov," and "On the General Theory of Control System." In the 1960 Moscow Congress Program, there are papers entitled "Information Flow Criteria for Feedback Control Systems," "Theory of Nonlinear Control," "Adaptive Control System Philosophy," and "The Philosophy of Control." Many of these papers discuss the advantages and disadvantages of feedback, the importance of the sensitivity concept, the control of information flow or energy conversion, the relationship between reliability, efficiency, and control loop configurations, the fundamental structure of control loops, and the deeper thoughts pertaining to memory, adaption, and learning.

What is the significance of the return to fundamental explanations of a field that has been growing almost exponentially in the last twenty years? Certainly, basic control principles have been understood during this period, but at the same time, they have been occasionally misinterpreted or misused, and it is a healthy sign, a sign of maturity perhaps, to note that control engineers are pausing to reflect on the fundamental knowledge that governs their sphere of activity. Some of the reflections are made verbally and others are expressed mathematically. Although fundamental in nature and sometimes inapplicable, thinking of this nature often leads to more useful concepts and broader applications.

The advance of control theory has been based on a combination of mathematical theory and its practical interpretation in constructing physical equipment. Future advances will undoubtedly be derived from applications of different mathematical innovations, and this issue of the TRANSACTIONS includes several papers which feature new applications of mathematics to the control field. Some of them may have no immediate application, but enterprising control engineers may find them useful in the near future. The excellent, thought-provoking paper by Bellman and Kalaba is a particular example. It stresses the underlying principles of control including feedback and adaptive processes, and it attempts to define them with mathematical terms as a foundation for future work in adaptive control. It is hoped that future TRANSACTIONS will feature some papers which utilize these mathematical tools in the development of new control systems.

—*The Editor.*

The Issue in Brief

As noted in the editorial, many of the papers in this issue are concerned with basic mathematical tools that may be applied to the field of automatic control. Also included is a discussion of the International Federation of Automatic Control, a second installment of a tutorial paper on vector quantities in control, and another bibliography on Time Lag Systems.

The International Federation of Automatic Control, **Harold Chestnut**

As explained in the previous issue,[1] the American Automatic Control Council (AACC) represents the United States in the International Federation of Automatic Control (IFAC), and the PGAC represents the IRE in the AACC. We are fortunate to have the first president of IFAC, Harold Chestnut, discuss the policies, progress, and significance of this international organization.

Dynamic Programming and Adaptive Processes: Mathematical Foundation, **R. Bellman and R. Kalaba**

This paper discusses fundamental aspects of control systems, the principles of feedback, and the nature of adaptive processes. The central purpose of the paper is to provide a foundation for the mathematical treatment of broad classes of the adaptive processes through the use of the concepts of dynamic programming.

The Properties and Methods for Computation of Exponentially-Mapped-Past Statistical Variables, **Joseph Otterman**

Various tools are used in statistical analyses and this paper discusses the properties and methods of computation with an exponential weighting function decreasing into the past. This weighting function results in ease of computation, and it provides simple theoretical relationships when the interest is focused on the recent behavior of the process involved. Analog computer circuits and digital computer flow diagrams which serve to compute the exponentially-mapped-past statistical variables are presented.

Generalized Weighting Function and Restricted Stability of a Linear Pulse-Modulated Error Feedback System, **William A. Janos**

An extensive mathematical development is used to produce a generalized weighting function, for a linear feedback system with a pulse-modulated error signal, in the form of a matrix operator acting on an input vector consisting of the first R-1 input derivatives where R is the order of the unmodulated closed-loop system. Since the system operator has the form of a finite dimensional matrix, it becomes possible to make and to realize more stringent conditions on the transient stability in terms of a preassigned bounded output after a preassigned time. Two examples are given to illustrate the procedure.

[1] R. Oldenburger, "The American Automatic Control Council," IRE Trans. on Automatic Control, vol. AC-4, pp. 5–7; May, 1959.

Optimization Based on a Square-Error Criterion with an Arbitrary Weighting Function, **G. J. Murphy and N. T. Bold**

In recent years many error criteria have been devised to measure system performance. This paper briefly reviews and compares most of these error criteria, then a new, generalized version of several methods is introduced, and it is referred to as the mean-weighted-square-error criterion. An optimum, realizable weighting function is derived by minimizing the error defined by the new criteria.

Multiple-Rate Sampled Data System, **Lester A. Gimpelson**

Large classes of digital systems employing computers have several pulse rates for economy or because of the nature of the computation. This paper discusses techniques for analyzing these systems containing both discrete and continuous variables with flow graphs and submultiple rate Z transforms. Tables and examples are given to illustrate the techniques involved.

Automatic Control of Three-Dimensional Vector Quantities—Part II, **Allen S. Lange**

This is the second part of a three-part tutorial article on the use of vector algebra in control systems. The first part considered the position vector problems encountered in spherical trigonometry and this part introduces the angular velocity vector for the purpose of analyzing and designing geometric stabilization systems.

Root Locus Properties and Sensitivity Relations in Control Systems, **Hanoch Ur**

Properties of the well-known root locus including pole sensitivity, angle of slope, and curvature are investigated in this paper, and a relationship between the sensitivity function and pole sensitivity is established. It is shown that the sensitivity determines variations in the transfer function due to large as well as small variations in the loop gain. Additional properties of the locus are also developed.

Time Lag Systems—A Bibliography, **N. H. Choksy**

Previous Transactions have included several bibliographies. This particular one is a large supplement to the one given in the May, 1959, issue[2] and it includes a thorough discussion of the mathematical characterization of time lags.

Correspondence

An interesting property of the root locus, often overlooked, is clearly explained by C. S. Lorens and R. C. Titsworth in their discussion on the "Properties of Root Locus Asymptotes."

PGAC News

Details of the progress being made in organizing the first IFAC technical meeting in Moscow, as reported by Professor Letov of the U.S.S.R., are given to indicate the complexity and difficulties that attend an international meeting of this size.

[2] R. Weiss, "Transportation lag—an annotated bibliography," IRE Trans. on Automatic Control, vol. AC-4, pp. 56–68; May, 1959.

The International Federation of Automatic Control*

HAROLD CHESTNUT†

Summary—**The initials IFAC, standing for the International Federation of Automatic Control, are appearing more and more in the control systems literature these days. It is appropriate that readers of these TRANSACTIONS be more fully aware of this organization in which they participate through the IRE's membership in the American Automatic Control Council. What are IFAC's purposes, who are its members, how does it operate, what has it accomplished? Now, after two years of existence for IFAC, these data are better known and have more significance.**

Purpose

As STATED in its Constitution, the purpose of IFAC is "to promote the science of automatic control." Automatic control is deemed to cover the field of open and closed loop (feedback) control of physical systems in theoretical and applied aspects.

"IFAC is to promote the science of automatic control by: 1) The interchange and circulation of information on automatic control activities in cooperation with national and other international organizations; 2) International Congresses; 3) Such other means as may be considered desirable as for instance publications."

Its purpose is to strengthen the national organizations and technical societies working in the field of automatic control by providing them with a way of working together on an international basis. It also provides a means for individual members of these societies to learn of their counterparts' activities in other countries and to establish contact and exchange technical information more directly.

Origin and Members

In a recent article,[1] Rufus Oldenburger has described the significant facts concerned with the founding of IFAC in Paris, France in September, 1957. IFAC is a federation of member organizations each representing the technical societies interested in automatic control for the country represented. "For each country, one scientific or professional engineering organization or one council formed by two or more such organizations having an interest in the field of automatic control shall be eligible for membership of IFAC."

On this basis, the following 22 nations are represented in IFAC by the member organizations and addresses listed.

Austria—Österreichischer Arbeitsausschuss für Automatisierung, Österreichisches Produktivitäts-Zentrum, Hohenstaufengasse 3, Wien 1.

Belgium—Institut Belge de Régulation et d'Automatisme, 3, rue Ravenstein, Bruxelles.

China—Chinese Association of Automation, c/o Institute of Automation, Academia Sinica, Peking.

Czechoslovakia—Laborator pro Automatisaci a Telemechaniku, Cekoslovenska Akademie Ved, Zitna 28, Praha.

Denmark—Dansk Ingeniorforening, 29, Vester Farimagsgade, Copenhagen.

Finland—Industrial Instrumentation and Control Engineering Professional Group, c/o EKONO, E. Espl. 14, Helsinki.

France—Association Française de Régulation et d'Automatisme (AFRA), 19, Rue Blanche, Paris.

Hungary—Hungarian National Committee of IFAC, c/o Budapesti Muszaki Egyetem, Villamosgepek Uzemtana Tanszek, Stoczek-u. 2 T.-EP. 111. 36, Budapest.

India—The Institution of Engineers, 8 Gokhale Road, Calcutta.

Israel—Ministry of Defense, Scientific Department, Hakirya, P.O. Box 7057, Tel-Aviv.

Italy—Consiglio Nazionale delle Ricerche, Ufficio relazioni internazionali, Piazzale delle Scienze 7, Roma.

Japan—National Committee of Automatic Control, Science Council of Japan, Ueno Park, Tokyo.

Netherlands—Koninklijk Instituut van Ingenieurs, Sectie voor Regeltechniek, Prinsessegracht 23, Den Haag.

Norway—Norsk Forening for Automatisering, Kranprinsensgate 17, Oslo.

Poland—Polski Komitet Automatyki, Naczelna Organizacja Techniczna, ul. Czackiego 3/5, Warszawa.

Roumania—Comisia de Automatizare, Academia Republicii Populare Romine, Calea Victoriei 125, Bucharest.

Sweden—Svenska Centralcommitten for internationella ingenjorskongresser, c/o Svenska Teknologforeningen, Brunkebergstorg 20, Stockholm.

Switzerland—Schweizerische Gesellschaft für Automatik, Sternwartstr. 7, Zurich.

Turkey—Teknik Universitesi, Rektorlugu, Istanbul.

Union of Soviet Socialist Republics—Nazionalny Komitet SSSR po Avtomaticheskomu Upravleniu, Kalantschevskaia ulitza 15a, Moskva.

United Kingdom—British Conference on Automation and Computation, c/o Institution of Electrical Engineers, Savoy Place, London W.C. 2.

United States of America—American Automatic Control Council, c/o Mr. W. E. Vannah, Secretary, 330 West 42nd Street, New York 36, N. Y.

Method of Operating

The business of IFAC is conducted by its General Assembly, its Executive Council, and its Committees as described below.

General Assembly

The General Assembly is the supreme body of the federation and consists of delegations of the member organizations, each member organization being entitled to one vote. It meets infrequently, generally upon the occasion of an international congress of IFAC. An extraordinary meeting of the General Assembly was held in Chicago on September 16–18 to review plans for the First International Congress of IFAC in Moscow, in 1960, to elect officers and to modify the Constitution and Bylaws.

* Manuscript received by the PGAC, October 22, 1959.

† General Electric Co., Schenectady N. Y.

[1] R. Oldenburger, "The American Automatic Control Council," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-4, pp. 5–7; May, 1959.

Executive Council

The management of the federation is vested in an Executive Council consisting of the President, the First and Second Vice-Presidents, the Treasurer, six Ordinary Members, and the immediate Past President. In addition, serving the Executive Committee and appointed by it are the Secretary and the Editor. The present incumbents of these offices are listed below. The Executive Council has been meeting annually to conduct the necessary business of directing IFAC.

President—A. M. Letov, Russia.
First Vice-President—E. C. Gerecke, Switzerland.
Second Vice-President—O. Benedikt, Hungary.
Treasurer—M. Cuenod, Switzerland.
Ordinary Members—J. F. Coales, United Kingdom; G. Evangelisti, Italy; P. J. Nowacki, Poland; J. Balchen, Norway; K. Kaneshigi, Japan; Z. Trnka, Czechoslovakia.
Past President—H. Chestnut, United States.
Secretary—G. Ruppel, Germany.
Editor—V. Broida, France.

Committees

The Executive Council is assisted in its work by an Advisory Committee and is entitled to establish Technical and Special Committees to deal with special subjects. The Advisory Committee is international in nature and consists of not more than one representative from each member organization. It has the duty of advising the Executive Council regarding the technical work of IFAC. Also the Advisory Committee shall recommend Technical and Special committees as required, specifying their areas of work. D. P. Eckman, United States, is the Chairman of the Advisory Committee and J. Loeb, France, is Vice-Chairman.

The following Technical Committees have been established with Chairmen at the addresses noted.

Applications—J. Mozley, Johns Hopkins University, Baltimore, Maryland.
Bibliography—W. Oppelt, University of Darmstadt, Darmstadt, Germany.
Components—G. Boromissza, Villamosgepek Uezemtant Tanszek, Budapesti Muszaki Egyetem, Stozcek-u. 2, Budapest XI, Hungary.
Education—A. Marino, Consiglio Nazionale delle Ricerche, Piazzale delle Scienze, 7, Rome, Italy.
Terminology—E. C. Gerecke, ETH, Schweizerische Gesellschaft für Automatik, Sternwartstr. 7, Zurich 6, Switzerland.
Theory—B. N. Petrov, Institute of Automatics and Telemechanics, Academy of Science, 9 leningrader Chaussee, Moscow D40, USSR.

AACC representatives to these committees include:

Bibliography—T. J. Higgins, University of Wisconsin, Madison, Wisconsin.
Components—D. J. Pompeo, Shell Development Corporation, Emeryville, California.
Education—Otto J. Smith, University of California, Berkeley, California.
Terminology—H. L. Mason, National Bureau of Standards, Washington 25, D. C.
Theory—John G. Truxal, Brooklyn Polytechnic Institute, Brooklyn, New York.

The functions of these committees in their respective areas include the preparation of programs of activity of service in the field of automatic control on an international basis. Included in these programs in the future will be the preparations for Congresses and other special meetings. In addition, these committees will prepare periodic reviews of the technical state of the art in automatic control and associated fields. At present, detailed programs and plans are being prepared by the various committees with reports of progress to be presented at the Moscow Congress in June, 1960.

A special committee chaired by Professor C. J. D. M. Verhagen was formed to present for IFAC a program on Instruments and Transducers in Automatic Control at the recent ISA Conference and Exhibit at Chicago in September, 1959. These presentations by nine speakers were well received and served to show how helpful international meetings can be from the viewpoint of speaker and audience alike. Although the Technical and Special Committees should be small for effective action, each member organization may recommend members to the chairmen of these committees. The members of each committee are selected by the committee chairmen with the concurrence of the Advisory Committee. In addition, these committees may invite other experts to offer their services in particular areas, as for example, on specific subcommittees.

Present Status and Future Outlook

IFAC has grown in membership and strength during its two years of existence and looks forward to a future of continued growth and greater effectiveness. Five issues of the IFAC bulletin have been published and continued publication on a three or four issues per year basis is planned by Victor Broida of France, the Editor. From the bulletin, engineers and scientists are obtaining an increasing awareness of what is going on internationally in the field of automatic control. Publication in the bulletin by more member organizations is anticipated, and more letters on technical matters by individuals will be included.

International exchange visits of experts fron both industry and university have been made between a number of member organizations in IFAC and further arrangements are being planned for student and professor exchanges in the future. These exchanges should speed the development of new ideas more rapidly than heretofore.

Foremost in IFAC's accomplishments is the work being done in preparation for the Moscow Congress reported elsewhere in this issue. The work of the National Committee of the USSR for Automatic Control has been thorough and well organized. The completeness of these preparations stresses the fact that everything possible is being done to make this Congress an outstanding one. In addition, the work of paper solicitation and review being done by the various member organizations is in many cases a major undertaking. Arrangements for

preparation of an English version of the preprints and publication of the proceedings have been made with Butterworths Publishing Limited by Professors John F. Coales of England and John R. Ragazzini of the United States as co-editors.

IFAC has established a sound working organization and a cordial relationship among individuals in its many member organizations. IFAC has provided a means whereby the science of automatic control can grow and flourish more rapidly and effectively. With its newly modified Constitution and Bylaws, its fine plans for the Moscow Congress in 1960, and its sound foundation of committees, IFAC can look forward with confidence for its new officers to carry it to a more successful future.

Automatic Control can play an important role in helping to bring about greater productivity and a better world in which to live. IFAC can help engineers and scientists in America as well as throughout the world to fulfill their part in making automatic control more effective.

Dynamic Programming and Adaptive Processes: Mathematical Foundation*

R. BELLMAN† AND R. KALABA†

Summary—**In many engineering, economic, biological, and statistical control processes, a decision-making device is called upon to perform under various conditions of uncertainty regarding underlying physical processes. These conditions range from complete knowledge to total ignorance. As the process unfolds, additional information may become available to the controlling element, which then has the possibility of "learning" to improve its performance based upon experience; *i.e.*, the controlling element may *adapt* itself to its environment.**

On a grand scale, situations of this type occur in the development of physical theories throgh the mutual interplay of experimentation and theory; on a smaller scale they occur in connection with the design of learning servomechanisms and adaptive filters.

The central purpose of this paper is to lay a foundation for the mathematical treatment of broad classes of such *adaptive processes*. This is accomplished through use of the concepts of dynamic programming.

Subsequent papers will be devoted to specific applications in different fields and various theoretical extensions.

I. Introduction

THE PURPOSE of this paper is to lay a foundation for a mathematical theory of a significant class of decision processes which have not as yet been studied in any generality. These processes, which will be described in some detail below, we shall call *adaptive.*

They arise in practically all parts of statistical study, practically engulf the field of operations research, and play a paramount role in the current theory of stochastic control processes of electronic and mechanical origin. All three of these domains merge in the consideration of the problems of communication theory.

Independently, theories governing the treatment of processes of this nature are essential for the understanding and development of automata and of machines that "learn."

We propose to illustrate how the theory of dynamic programming [1] can be used to formulate in precise terms a number of the complex and vexing questions that arise in these studies. Furthermore, the functional equation approach of dynamic programming enables us to treat some of these problems by analytic means and to resolve others, where direct analysis is stymied, by computational techniques.

In this paper, general questions are treated in an abstract fashion. In subsequent papers, we shall apply the formal structure erected here to specific applications.

II. Adaptive Processes

We wish to study multi-stage decision processes, and processes which can be construed to be of this nature, for which we do not possess complete information. This lack of information takes various forms of which the following are typical.

We may not be in possession of the entire set of admissible decisions; we may not know the effects of these decisions; we may not be aware of the duration of the processes and we may not even know the over-all purpose of the process. In any number of processes occurring in the real world, these are some of the difficulties we face.

The basic problem is that of making decisions on the basis of the information that we do possess. An essential part of the problem is that of using this accumulated

* Manuscript received by the PGAC, September 9, 1958; revised manuscript received, February 12, 1959.
† The RAND Corporation, Santa Monica, Calif.

knowledge to gain further insight into the structure of the processes, using analytic, computational, and experimental techniques.

From this intuitive description of the types of problems that we wish to consider, it is clear that we are impinging upon some of the fundamental areas of scientific research. Obvious as the existence of these problems are, it is not at all clear how questions of this nature can be formulated in precise terms.

Particular processes of this type have been treated in a number of sources, such as the works on sequential analysis [14]; the theory of games [13]; the theory of multi-stage games;[1] and papers on "learning processes" [2], [3], [5]–[8], [11].

III. The Unfolding of a Physical Process

In order to appreciate the type of process we wish to consider, the problems we shall treat, the terminology we shall employ, and the methods we shall use, it is essential that we discuss, albeit in abstract terms, the behavior of the conventional deterministic physical system.

Let a system S be described at any time t by a state vector p. Let $t_1, t_2, \cdots$, be a sequence of times, $t_1 < t_2 < \cdots$, at which the system is subject to a change which manifests itself in the form of a transformation. At time t_1, p_1 is converted into $T_1(p_1)$, at time t_2, $p_2 = T_1(p_1)$ is converted into $T_2(p_2)$, and so on, with the result that the sequence of states of the system is given by the sequence $\{p_k\}$, where

$$p_{k+1} = T_k(p_k), \qquad k = 1, 2, \cdots. \tag{1}$$

The state of the system at the end of time t_N is then given by

$$p_{N+1} = T_N(T_{N-1}(\cdots T_2(T_1(p_1))\cdots)), \tag{2}$$

where p_1 is the initial state of S.

If $T_k(p)$ is independent of k, which is to say, if the same transformation is applied repeatedly, then the preceding result can be written symbolically in the form

$$p_{N+1} = T^N(p_1). \tag{3}$$

The interpretation of the behavior of a physical system over time as the iteration of a transformation was introduced by Poincaré, and extensively studied by G. D. Birkhoff [4] and others. It furnishes the background for the application of modern abstract operator theory to the study of physical systems, as, for example, in quantum mechanics [12]. The idea of using this fundamental representation in connection with the formulation of the ergodic theorem is due to B. O. Koopman.

IV. Feedback Control

With all this in mind, we are now able to introduce the concept of *feedback control.*

Supposing that the behavior of the system as described by the foregoing equations is not satisfactory, we propose to modify it by changing the character of the transformation acting upon p. This change will be made dependent upon the state of the system at the particular time the transformation is applied.

In order to indicate the fact that we now have a choice of transformations, we write $T(p, q)$ in place of p. The variable q indicates the choice that is made. Consequently, we shall call it the *control variable*, as opposed to p, the *state variable*. To simplify the notation and discussion, we shall assume that the set of admissible transformations does not vary with time.

If q_k denotes the choice of the control variable at time t_k, we have, in place of (1), the relation

$$p_{k+1} = T(p_k, q_k), \qquad k = 1, 2, \cdots, \tag{4}$$

with p_{N+1} explicitly determined as in (2).

The associated variational problem is that of choosing $q_1, q_2, \cdots, q_N$ so as to make the behavior of the system conform as closely as possible to some preassigned pattern. We wish, however, to do more than leave the problem in this vague format.

V. Causality

Turning back, from the moment, to the deterministic, uncontrolled process discussed in Section III, let us note that the state of the system at time t_{k+1} is a function of the initial state of the system and the number of transformations that have been applied. Consequently, we may write

$$p_{k+1} = f_k(p_1), \tag{5}$$

where p_1 is the initial state of the system.

For the sake of convenience, let us merely write p in place of p_1. Then, the function $f_k(p)$ is easily seen to satisfy the basic functional equation

$$f_{m+n}(p) = f_m(f_n(p)), \qquad m, n = 1, 2, \cdots. \tag{6}$$

This is the fundamental semigroup property of dynamical systems.

VI. Optimality

With the foregoing as a guide, let us see if we can formulate the feedback control process in the same terms.

To illustrate the applicability of the functional equation technique, let us consider a finite process, of N stages, where it is desired to maximize a preassigned function, ϕ, of the final state of the system, p_N. This is often called a *terminal control* process.

The variational problem may now be posed in the following terms:

$$\underset{[q_1, q_2, \cdots, q_N]}{\text{Max}} \ \phi(p_N). \tag{7}$$

This maximum, which we shall assume exists, is again a function of the initial state, p, and the duration of the

[1] See R. Bellman [1], ch. 10.

process. Let us then introduce the function defined for all states p and $N=1, 2, \cdots$, by the relation

$$f_N(p) = \operatorname*{Max}_{q} \phi(p_N), \tag{8}$$

where q represents the set $[q_1, q_2, \cdots, q_N])$.

Let us now introduce some additional terminology. A set of admissible choices of the q_i, $[q_1, q_2, \cdots, q_N]$, will be called a *policy*; a policy which maximizes $\phi(p_{N+1})$ will be called an *optimal policy*.

In order to obtain a functional equation corresponding to (6), we invoke the

PRINCIPLE OF OPTIMALITY: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The mathematical transliteration of this statement is the functional relation

$$f_N(p) = \operatorname*{Max}_{q_1} f_{N-1}(T(p, q_1)), \tag{9}$$

$N=2, 3, \cdots$, with

$$f_1(p) = \operatorname*{Max}_{q_1} \phi(T(p, q_1)). \tag{10}$$

Further discussion, and various existence and uniqueness theorems for the functions $\{f_i(p)\}$ and the associated policies will be found in [1].

In this way, the calculus of variations is seen to be a part of an extension of the classical theory of iteration, and of semigroup theory.

VII. Stochastic Elements

In order to treat questions arising in the physical world in precise fashion, it is always necessary to make certain idealizations. Foremost among these is the assumption of known cause and effect, and perhaps, even that of cause and effect in itself.

To treat physical processes in a more realistic way, we must take into account unknown causes and unknown effects. We find ourselves in the ironical position of making precise what we mean by ignorance.

At the present time, there exist a number of approaches to this fundamental conundrum, all based upon the concept of a random variable. Building upon this foundation is the theory of games.

We shall discuss here only the direct application of the concept of stochastic processes, leaving the game aspects for a later date.

The theory of probability in a most ingenious fashion skirts the forbidden region of the unknown by ascribing to an unknown quantity a distribution of values according to certain law. Having taken this bold step, it is further agreed that we shall measure performance not in terms of a single outcome, but in terms of an average taken over this distribution of values. Needless to add, this artifice has been amazingly successful in the analysis of physical processes; *e.g.*, statistical mechanics, quantum mechanics.

Following this line of thought, we begin to take account of unknown effects by supposing that the result of a decision q is not to transform p into a fixed state $T(p, q)$, but rather to transform p into a stochastic vector z whose distribution function is $dG(z, p, q)$, dependent upon both the initial vector p and the decision q. Let us further suppose that the purpose of the process is to maximize the expected value of a preassigned function, ϕ, of the final state of the system.

Before setting up the functional equation analogous to (9), let us review the course of the process. At the initial time, an initial decision q_1 is made, with the result that there is a new state p_1, which is observed. On the basis of this information, a new decision, q_2, is made, and so on.

It is important to emphasize the great difference between a feedback control process of this type, in which the q_i are chosen stage-by-stage, and process in which the q_i are chosen all at once, at some initial time.

In the deterministic case, the two processes are equivalent, and it is only a matter of convenience whether we use one or the other formulation.[2] In the stochastic case, the two processes are equivalent only in certain special situations. We shall be concerned here only with the stage-by-stage choice.

The analog of (10) is then

$$f_1(p) = \operatorname*{Max}_{q} \int_z \phi(z) dG(z, p, q), \tag{11}$$

and that of (9) is

$$f_N(p) = \operatorname*{Max}_{q} \int_z f_{N-1}(z) dG(z, p, q), \quad N = 2, 3, \cdots.^3 \tag{12}$$

This type of process has been discussed in some detail [1].

VIII. Second Level Processes

Fortunately for the mathematician interested in these processes, the tale does not end here! For in a number of significant applications, it cannot be safely assumed that the unknown quantities possess known distribution functions.

In many cases, we must face the fact that we are dealing with more complex situations in which far less is known about the unknown quantities. For a discussion of the importance of these processes in the general theory of design and control, see McMillan [9]; for a discussion of the dangers and difficulties inherent in *any* mathematical treatment, see Zadeh [15].

[2] This corresponds to the choice we have of describing a curve as a locus of points or as an envelope of tangents.

[3] The descriptive version of this equation, when no control is exerted, is, of course, the Chapman-Kolmogoroff equation, the stochastic analog of (6).

A first attempt in salvaging much of the structure already erected is to assume that the unknown quantities possess fixed, but unknown, distribution functions. Regarding deterministic processes as those of zeroth level, and the stochastic processes described in Section VII as first-level processes, we shall refer to these new stochastic processes as *second-level* processes.

Although it is clear that we now possess a systematic method for constructing a hierarchy of mathematical models, we shall restrain ourselves in the remainder of this paper to the discussion of second-level processes.

IX. Additional Assumptions

Some further assumptions are required if we wish to proceed from this point to an analytic treatment. These are

1) We possess an *a priori* estimate for the distribution function governing the physical state of the system, which, until further knowledge is acquired, we regard as the actual distribution.
2) We possess a set of rules which tells us how to modify this *a priori* distribution so as to obtain an *a posteriori* distribution when additional information is obtained.
3) We possess an *a priori* estimate for the distribution functions governing the outcomes of decisions, which, until further knowledge is acquired, we regard as the actual distribution, and, as above, we know how to modify this in the light of subsequent information.

In this paper, we restrict ourselves to the case of known physical states.

In formal terms, our state vector is now compounded of a point in phase space p and an *information pattern*, $dG(z, p, q)$. As a result of a decision q_1, there result the transformations

$$\begin{aligned} p_0 &\rightarrow p_1 && \text{(observed)} \\ dG(z, p^*, q) &\rightarrow dH(z, p^*, q; p_0, G, q_1, p_1) && \text{(hypothesized)}. \quad (13) \end{aligned}$$

On the basis of these assumptions, and considering a control process which continues in time as described in Section VII, we wish to pose the problem of determining optimal policies. For the first time, we are considering adaptive processes significantly different from those of the usual deterministic or stochastic control process.

X. Functional Equations for Second-Level Processes

As before, we introduce the function

$f_N(p; G(z, p^*, q))$ = the expected value of $\phi(p_N, G_N)$ (14) obtained using an optimal policy for an N-stage process starting in state (p, G).

Depending upon the objectives of the process, only one or the other of p_N and G_N may enter into ϕ. Examples of both extremes abound.

Arguing as in the preceding sections, we see that the basic recurrence relation is

$$f_N(p; G(z, p^*, q)) = \operatorname*{Max}_{q_1} \int_w f_{N-1}(w; H(z, p^*, q; p, G, q_1, w)) dG(w, p, q_1), \quad (15)$$

for $N=2, 3, \cdots$, with

$$f_1(p; G(z, p^*, q)) = \operatorname*{Max}_{q_1} \int \phi(w, H(z, p^*, q; p, G, q_1, w)) dG(w, p, q_1). \quad (16)$$

These equations are quite useful in the derivation of existence and uniqueness theorems concerning optimal policies, return functions, and in ascertaining certain structural properties of optimal policies [1], [2].

If, however, we treat processes which are too complex for a direct analytic approach, as is invariably the case for realistic models, we wish to be able to fall back upon a computational solution. The occurrence of functions of functions, *e.g.* the sequence $\{f_N(p; G)\}$, effectively prevents this.

XI. Further Structural Assumptions

In order to reduce the foregoing equations to more manageable form, let us assume that the structure of the actual distribution is known, but that the uncertainty arises with regard to the values of certain parameters.

At any stage of the process, in place of an *a priori* estimate, $G(z, p, q)$, for the distribution function, we suppose that we have an *a priori* estimate for the distribution function governing the unknown parameters. Again, a basic assumption is that this distribution function exists.

The functional equations that we derive are exactly as above, with the difference in meaning of the distribution functions that we have just described.

XII. Reduction from Functionals to Functions

We are now ready to take the decisive step of reducing $f_N(p, G)$ from a functional to a function.

It may happen, and we will give an example, that the change in the distribution function, from $G(z, p, q)$ to $H(z, p^*, q; p, G, q_1, w)$ is one that can be represented by a point transformation. This will be the case if G and H are both members of a family of distribution functions $K(z; \alpha)$ characterized by a vector parameter α. Thus, if

$$\begin{aligned} G(z, p, q) &\equiv K(z, p, q; \alpha) \\ H(z, p^*, q; p, G, q_1, w) &\equiv K(z, p, q; \beta), \end{aligned} \quad (17)$$

the change from G to H may be represented by

$$\beta = \psi(p, \alpha, q_1, w). \quad (18)$$

Then we may write

$$f_N(p, G(z, p, q)) \equiv f_N(p; \alpha), \quad (19)$$

and (15) becomes

$$f_N(p; \alpha) = \underset{q_1}{\text{Max}} \int_w f_{N-1}(w; \beta) dK(w; \alpha). \tag{20}$$

The dependence upon q_1 is by way of (18).

XIII. An Illustrative Process—Deterministic Version

Let us now show how these ideas may be applied to the study of control processes. Consider a discrete scalar recurrence relation of the form

$$u_{n+1} = au_n + v_n, \qquad u_0 = c. \tag{21}$$

Here u_n is the state variable and v_n is the control variable. Suppose that the sequence $\{v_n\}$ is to be chosen to minimize the function

$$|u_N| + b \sum_{k=1}^{N} u_k^2, \tag{22}$$

subject to the constraints

$$|v_i| \leq r, \qquad i = 0, \cdots, N-1. \tag{23}$$

Although the precise analytic form of the criterion function is of little import as far as the present discussion is concerned, we have used specific functions to make the presentation as concrete as possible. Furthermore, the defining equation need not be linear.

This is a simple example of a deterministic control process. Introduce the sequence of functions defined by the relation

$$f_N(c) = \underset{\{v_i\}}{\text{Min}} \left[|u_N| + b \sum_{k=1}^{N} u_k^2 \right], \tag{24}$$

where N takes on the values 1, 2, $\cdots$, and c any real value.

Then

$$f_1(c) = \underset{|v_0| \leq r}{\text{Min}} \left[|ac + v_0| + b(ac + v_0)^2 \right], \tag{25}$$

and for $N \geq 2$, the principle of optimality yields the relation

$$f_N(c) = \underset{|v_0| \leq r}{\text{Min}} \left[b(ac + v_0)^2 + f_{N-1}(ac + v_0) \right]. \tag{26}$$

XIV. Stochastic Version

In place of the recurrence relation of (21), let us introduce a stochastic transformation

$$u_{n+1} = au_n + r_n + v_n, \qquad u_0 = c. \tag{27}$$

Here $\{r_n\}$ is a sequence of independent random variables assuming only the values 1 and 0. Let

$$\begin{aligned} r_n &= 1 \quad \text{with probability } p \\ &= 0 \quad \text{with probability } 1 - p. \end{aligned} \tag{28}$$

The quantity p is known, and for simplicity taken to be independent of n, although this is not necessary.

We now wish to minimize the expected value of the quantity appearing in (22). This is now a stochastic control process of the type described above in general terms. Call the minimum expected value $f_N(c)$. Then, following the procedures of Section VII, we have the relations

$$\begin{aligned} f_1(c) &= \underset{|v_0| \leq r}{\text{Min}} \left[\int_{r_0} \left[|ac + v_0 + r_0| + b(ac + r_0 + v_0)^2 \right] dG(r_0) \right] \\ &= \underset{|v_0| \leq r}{\text{Min}} \Big[p\left[|ac + v_0 + 1| + b(ac + v_0 + 1)^2 \right] \\ &\quad + (1 - p)\left[|ac + v_0| + b(ac + v_0)^2 \right] \Big], \end{aligned} \tag{29}$$

and, for general N,

$$\begin{aligned} f_N(c) = \underset{|v_0| \leq r}{\text{Min}} \Big[& p\left[b(ac + v_0 + 1)^2 + f_{N-1}(ac + v_0 + 1) \right] \\ & + (1 - p)\left[b(ac + v_0)^2 + f_{N-1}(ac + v_0) \right] \Big]. \end{aligned} \tag{30}$$

XV. Adaptive Control Version

Let us now consider the adaptive control version. We are given the information that the random variables r_n possess distributions of the special type described above, but we do not know the precise value of p.

We shall assume, however, that we do possess an *a priori* distribution for the value of p, $dG(p)$, and that we possess a known rule for modifying this *a priori* distribution on the basis of the observations that are made as the process unfolds.

If we observe that over the past $m+n$ stages, the random variables have taken on m values of 1 and n values of 0, we take as our new *a priori* distribution the function

$$dG_{m,n}(p) = p^m (1 - p)^n dG(p) \Big/ \int_0^1 p^m (1 - p)^n dG(p), \tag{31}$$

a Bayes approach.[4]

Once we have fixed upon a choice of $G(p)$, the *a priori* distribution function at any stage of the process is uniquely determined from the foregoing by the numbers m and n. This simple observation enables us to reduce the information pattern from that of the specification of a number or vector, in general, plus a function $G_{m,n}(p)$, to that of the specification of three numbers, c and the two integers m and n.

In this way, we reduce the problem from one requiring the use of functionals to one utilizing only functions. This is an essential step not only for computational purposes, but for analytic purposes as well.

Let us then introduce the sequence of functions $\{f_N(c, m, n)\}$ defined once again as the minimum expected value of the quantity in (22), starting with the

[4] This is an assumption of the type called for in Section IX. Although reasonable, it is not the only one possible. There are analytical advantages in choosing G to be a beta distribution.

information pattern of m ones and n zeros, and state c. Then

$$f_1(c, m, n) = \operatorname*{Min}_{|v_0| \leq r} \Big[p_{m,n}[b(ac + v_0 + 1)^2 + |ac + v_0 + 1|] + (1 - p_{m,n})[b(ac + v_0)^2 + |ac + v_0|] \Big], \quad (32)$$

where $p_{m,n}$ is the expected probability using the probability distribution in (31), *i.e.*,

$$p_{m,n} = \frac{\int_0^1 p^{m+1}(1-p)^n dG(p)}{\int_0^1 p^m (1-p)^n dG(p)}. \quad (33)$$

For $N \geq 2$, we have the recurrence relation

$$f_N(c, m, n) = \operatorname*{Min}_{|v_0| \leq r} \Big[p_{m,n}[b(ac + v_0 + 1)^2 + f_{N-1}(ac - v_0 + 1, m + 1, n)] + (1 - p_{m,n})[b(ac + v_0)^2 + f_{N-1}(ac + v_0, m, n + 1)] \Big]. \quad (34)$$

In this fashion, we obtain a computational approach to processes with general criteria and an analytic approach to processes with criteria of particular type. A thoroughgoing discussion of the analytic aspects of the solution of processes of this nature described by linear equations and quadratic criteria will be found in a forthcoming doctoral thesis by Marshall Freimer. Previous applications of these techniques have been made in [2] and [3].

XVI. Sufficient Statistics

The fact that the past history of the process described in the preceding paragraphs can be compressed in the indicated fashion, so that functions rather than functionals occur, is a particular instance of the power of the theory of "sufficient statistics" [10].

Many further applications of this important concept will be found in the thesis by Freimer mentioned above.

In a number of cases, this compression of data occurs asymptotically as the process continues; *e.g.*, the central limit theorem. A number of quite interesting questions arise from this observation.

XVII. Discussion

In the foregoing pages, we have attempted to construct a mathematical foundation for the study of the many fascinating aspects of the field of adaptive control. In further papers, we shall discuss a number of complex problems which arise from this approach.

From the purely mathematical point of view, we are now able to contemplate a theory of continuous control processes of adaptive type, obtained as a limiting form of the theory of discrete control processes. A variety of significant convergence questions are encountered in this way.

Furthermore, we can construct a theory of multistage games on the same foundations.

Finally, the problem of computational solution is by no means routine, and there are a variety of interesting approaches based upon approximations in function space and approximations in policy space to be explored.

From the conceptual point of view, we must face the fact that there are many further uncertainties to be examined in the state of the system, in the observation of the random effect, in the transmission of the control signal, in the duration of the process, and even in the criterion function itself.

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The Properties and Methods for Computation of Exponentially-Mapped-Past Statistical Variables*

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Summary—**The exponentially-mapped-past (emp) statistical variables represent an approach to the statistical analysis of a process when the interest is focused on the recent behavior of the process. An exponential weighting function, decreasing into the past, in the case of continuously observed processes, and a geometric ratio, in the case of discrete data, are utilized. This approach is the simplest from the point of view of ease of computation, and at the same time it possesses the advantage of some simple theoretical relationships, which are discussed. Analog computer circuits and digital computer flow diagrams which serve to compute the exponentially-mapped-past statistical variables are presented.**

Introduction

THIS PAPER is concerned with an approach to the statistical analysis of a process when the interest is focused on the recent behavior of the process. This approach offers the advantage of simplicity in computation when the analysis is carried out by computers; and at the same time, it possesses the advantage of some simple theoretical relationships. It is an extension of the work reported earlier.[1] The material is primarily theoretical, but analog computer circuits and digital computer flow diagrams which serve to compute the exponentially-mapped-past statistical variables are presented. The computational advantages, which have been realized by others,[2] are pointed out.

The exponentially-mapped-past statistical variables (which will be referred to as emp variables) are quantities relating to a set of observations computed in such a way that the recent values of the observations contribute more strongly than the values observed in the more distant past. The relative weighting is a geometric ratio in the case of discrete (naturally discrete or sampled) data and an exponential function in the case of continuously observed functions. The emp variables are meant to apply to control or detection problems; therefore, information about the process up to the present time only is utilized. The emp variables themselves are functions of time.

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Continuous EMP Variables

EMP Average

The emp average, $\bar{f}_\alpha(0)$, of a continuously observed variable[3] $f(t)$ will be defined as

$$\bar{f}_\alpha(0) = \alpha \int_{-\infty}^{0} f(t) e^{\alpha t} dt, \tag{1}$$

where the past extends from 0 to $-\infty$, and $1/\alpha$ is a time constant for which the constancy of the regime can be effectively assumed. The coefficient α in front of the integral sign comes to effect a normalized weighting function, since

$$\int_{-\infty}^{0} e^{\alpha t} dt = 1/\alpha. \tag{2}$$

The concept of $\bar{f}_\alpha(t)$ is the average of $f(\tau)$, determined experimentally with information about $f(\tau)$ up to the time t only. For an arbitrary moment t (value negative) in the past, the emp average will be given by

$$\bar{f}_\alpha(t) = \alpha \int_{-\infty}^{t} f(\tau) e^{\alpha(\tau - t)} d\tau = \alpha e^{-\alpha t} \int_{-\infty}^{t} f(\tau) e^{\alpha \tau} d\tau. \tag{3}$$

Eq. (3) can be expressed in another way as

$$\bar{f}_\alpha(t) = \alpha \int_{-\infty}^{\infty} f(\tau) e^{\alpha(t-\tau)} u(t - \tau) d\tau, \tag{4}$$

where $u(t)$ is a unit step function $[u(t) = 1$ for $t \geq 0$, $u(t) = 0$ for $t < 0]$. This is a convolution integral of $f(\tau)$ and an impulse response of a low-pass RC filter, or a "leaky" integrator, with leakage time constant $RC = 1/\alpha$. The computation of the emp average in accordance with (3) or (4) is shown in Fig. 1.

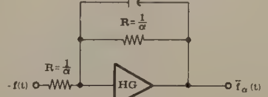

Fig. 1—Computation of $\bar{f}_\alpha(t)$ in accordance with (3).

The above definitions apply in the case in which observations extending indefinitely into the past can be assumed; *i.e.*, it is not necessary to have a good estimate

[3] The observed variable itself need not be continuous, but can assume discrete values only.

of the average during the initial stages of the analysis or control process. If such is not the case, we have (T designates the beginning of the observation relative to the present and is negative)

$$\bar{f}_\alpha(0, T) = \frac{\alpha}{1 - e^{\alpha T}} \int_T^0 f(\tau) e^{\alpha\tau} d\tau, \tag{5}$$

and for an arbitrary time in the past

$$\bar{f}_\alpha(t, T) = \frac{\alpha}{e^{\alpha t} - e^{\alpha T}} \int_T^t f(\tau) e^{\alpha\tau} f\tau. \tag{6}$$

This definition is adopted, since

$$\int_T^t e^{\alpha\tau} d\tau = \frac{e^{\alpha t} - e^{\alpha T}}{\alpha}. \tag{7}$$

It follows that if (3) is used instead of the more accurate (5), in the initial stages of observation when e^T is not negligible the absolute value of the emp average will be smaller than the "true" value, *i.e.*, the value resulting from the use of (5). The computation of the emp average in accordance with (5) is shown in Fig. 2.

EMP Variance

The emp variance $s_\alpha^2(0)$ is defined as follows:

$$s_\alpha^2(0) = \alpha \int_{-\infty}^0 [f(t) - \bar{f}_\alpha(t)]^2 e^{\alpha t} dt, \tag{8}$$

and for time t in the past the emp variance will be

$$s_\alpha^2(t) = \alpha \int_{-\infty}^t [f(\tau) - \bar{f}_\alpha(\tau)]^2 e^{\alpha(\tau - t)} d\tau$$

$$= \alpha e^{-\alpha t} \int_{-\infty}^t [f(\tau) - \bar{f}_\alpha(\tau)]^2 e^{\alpha\tau} d\tau. \tag{9}$$

The scheme for computing $s_\alpha^2(t)$ in accordance with this equation is shown in Fig. 3. This is based on the assumption that accurate computation in the initial stages is unimportant. If accurate computation at this stage is necessary, then the emp variance will be defined as

$$s_\alpha^2(t, T) = \frac{\alpha}{(e^{\alpha t} - e^{\alpha T})} \int_T^t [f(\tau) - \bar{f}_\alpha(\tau)]^2 e^{\alpha\tau} d\tau, \tag{10}$$

as was done for the average in (5). For the remainder of the section, (8) only will be used as defining the emp variance. Introducing the value of $\bar{f}_\alpha(t)$ as defined by (3) into (8) we obtain

$$s_\alpha^2(0) = \alpha \int_{-\infty}^0 \left[f(t) - \alpha e^{-\alpha t} \int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2 e^{\alpha t} dt. \tag{11}$$

After expanding the right side of the above equation,

$$s_\alpha^2(0) = \alpha \int_{-\infty}^0 f^2(t) e^{\alpha t} dt - 2\alpha^2 \int_{-\infty}^0 f(t) dt \int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau$$

$$+ \alpha^3 \int_{-\infty}^0 e^{-\alpha t} dt \left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2. \tag{12}$$

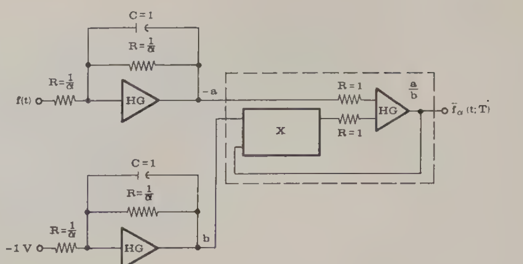

Fig. 2—Computation of $\bar{f}_\alpha(t, T)$ in accordance with (5).

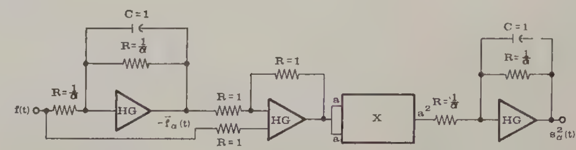

Fig. 3—Computation of $s_\alpha^2(t)$ in accordance with (8).

Theorem: A simple relationship, parallel to one existing in ordinary statistical parameters, will be now derived. The relationship involves the variable $\overline{f_\alpha^2}(0)$, which is defined as

$$\overline{f_\alpha^2}(0) = \alpha \int_{-\infty}^0 f^2(t) e^{\alpha t} dt. \tag{13}$$

It will be shown that for $s_\alpha^2(0)$, as defined by (8), the following holds:

$$s_\alpha^2(0) = \overline{f_\alpha^2}(0) - \bar{f}_\alpha^2(0). \tag{14}$$

To prove this, the last expression in (12) is integrated by parts. The integrand is regarded as a product of

$$\left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2$$

and $e^{-\alpha t}$:

$$\int_{-\infty}^0 e^{-\alpha t} dt \left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2$$

$$= -\frac{e^{-\alpha t}}{\alpha} \left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2 \Bigg|_{-\infty}^0$$

$$+ \frac{1}{\alpha} \int_{-\infty}^0 e^{-\alpha t} dt \frac{d}{dt} \left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2$$

$$= -\frac{1}{\alpha} \left[\int_{-\infty}^0 f(\tau) e^{\alpha\tau} d\tau \right]^2$$

$$+ \frac{2}{\alpha} \int_{-\infty}^0 e^{-\alpha t} f(t) e^{\alpha t} \int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau. \tag{15}$$

The above holds, since the lower limit, $t \to -\infty$, of the expression

$$-\frac{e^{-\alpha t}}{\alpha} \left[\int_{-\infty}^t f(\tau) e^{\alpha\tau} d\tau \right]^2$$

vanishes, if $f(\tau)$ has the property of any measured variable, *viz.*, that there exists a number A such that $|f(\tau)| < A$. Eq. (12) can be rewritten thus:

$$s_\alpha^2(0) = \alpha \int_{-\infty}^{0} f^2(t) e^{\alpha t} dt - 2\alpha^2 \int_{-\infty}^{0} f(t) dt \int_{-\infty}^{t} f(\tau) e^{\alpha\tau} d\tau$$

$$+ \alpha^3 \int_{-\infty}^{0} e^{-\alpha t} dt \left[\int_{-\infty}^{t} f(\tau) e^{\alpha\tau} d\tau \right]^2$$

$$= \alpha \int_{-\infty}^{0} f^2(\tau) e^{\alpha t} dt - 2\alpha^2 \int_{-\infty}^{0} f(t) dt \int_{-\infty}^{t} f(\tau) e^{\alpha\tau} d\tau$$

$$- \frac{\alpha^3}{\alpha} \left[\int_{-\infty}^{0} f(\tau) e^{\alpha\tau} d\tau \right]^2$$

$$+ \frac{2\alpha^3}{\alpha} \int_{-\infty}^{0} f(t) dt \int_{-\infty}^{t} f(\tau) e^{\alpha\tau} d\tau$$

$$= \overline{f_\alpha^2}(0) - \bar{f}_\alpha^2(0), \tag{16}$$

which constitutes the desired result. The scheme for computation of $s_\alpha(t)$ in accordance with (16) is shown in Fig. 4.

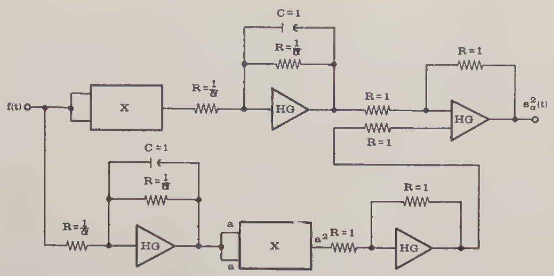

Fig. 4—Computation of $s_\alpha^2(t)$ in accordance with (16).

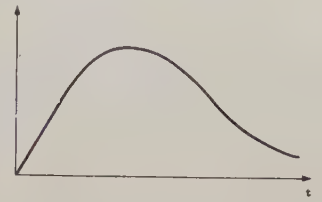

Fig. 5—Aperture in the form $e^{\alpha_1 t} - e^{\alpha_2 t}$ (t is negative).

EMP Fourier Transforms

Fourier transform $F(j\omega)$ of $f(t)$ represents an analysis of the presence of different frequency components in the phenomenon under observation.

The emp Fourier transform is defined as follows for an arbitrary moment t in the past:

$$F_\alpha(j\omega) = \alpha \int_{-\infty}^{t} f(x) e^{-\alpha(t-x)} e^{-j\omega x} dx. \tag{17}$$

The extension of the integration of $+\infty$ is permissible if $f(t)$ is assumed to vanish for $x > t$, *i.e.*, for the future relative to the time of interest. Thus,

$$F_\alpha(j\omega) = \alpha \int_{-\infty}^{\infty} f(x) e^{-\alpha(t-x)} u(t-x) e^{-j\omega x} dx, \tag{18}$$

where $u(t)$ is a step function described before.

Examination of the above equations shows that

$$F_\alpha(j\omega) = \alpha F(-\alpha + j\omega); \tag{19}$$

i.e., the emp. Fourier transform is not new but is the ordinary Fourier transform evaluated at a line, $-\alpha + j\omega$, parallel to the $j\omega$ axis, and multiplied by α. But the presentation here brings out the fact that $\alpha F(-\alpha + j\omega)$ represents an analysis for recent frequency-wise behavior of the observed variable, the "recentness" increasing with increasing α. It will be noted that $F_\alpha(0) = \bar{f}_\alpha(0)$; thus the emp average is equal to the dc ($\omega = 0$) value of the emp Fourier transform.

It may be noted that, if it is desired to concentrate on analysis in a certain region in the past, *i.e.*, to view the past through an aperture $e^{\alpha_1 t} - e^{\alpha_2 t}$ (plotted for $\alpha_2 = 2\alpha_1$ in Fig. 5), then the frequency analysis is given by $F_{\alpha_1}(j\omega) - F_{\alpha_2}(j\omega)$. This statement actually will apply to other emp variables, such as the emp average or the emp variance. The problem outlined in this paragraph is rather different from the question of the determination of the best recent analysis of the observed variable, for it is more related to interpolation than to extrapolation. Hence, it will not be discussed further in this paper.

EMP Density of Occurrence[4]

Let $f(t)$ be a random variable under observation. Assuming a stationary and ergodic process, let us concern ourselves with the probability, experimentally determined, that $f(t)$ lies between two values; or more specifically, the probability that

$$\left(f_1 - \frac{\Delta f}{2}\right) < f(t) < \left(f_1 + \frac{\Delta f}{2}\right).$$

This probability will be denoted as $p(f_1)\Delta f$. The important question of whether a limit for $p(f_1)$ exists as $\Delta f \to 0$ need not concern us in this discussion.

This probability can be defined on the basis of observation of $f(t)$ during a time interval T in the following manner:

$$p(f_1)\Delta f = \frac{\int_T^0 \theta(t) dt}{T}, \tag{20}$$

in which $\theta(t) = 1$ if

$$\left(f_1 - \frac{\Delta}{2}\right) < f(t) < \left(f_1 + \frac{\Delta f}{2}\right),$$

and $\theta(t) = 0$ elsewhere.

[4] The use of the word "frequency" in the statistical sense will be avoided in this paper.

A parallel definition of emp density of occurrence $p_\alpha(f_1)$ in a slowly changing process is first made for an infinite interval of observation (in (20), $T\to\infty$), thus:

$$p_\alpha(f_1)\Delta f = \alpha\int_{-\infty}^{0}\theta(t)e^{\alpha t}dt, \tag{21}$$

in which $\theta(t)$ is defined as before. When summation of the density of occurrence is carried out over all the possible range of $f(t)$, without overlapping of the intervals, the summation will result in unity. Thus, $p_\alpha(f_1)\Delta f$ has the basic property of a probability function, in that it ranges from zero to unity. Its physical significance is obvious from its mathematical definition: the experimentally determined density of occurrence is computed in such a way that the recent interval during which the observed function lies between $(f_1-\Delta f/2)$ and $(f_1+\Delta f/2)$ contributes more strongly than a similar interval in the more distant past.

When an infinite interval of observation cannot be assumed, the emp density of occurrence will be computed according to the formula

$$p_\alpha(f_1)\Delta f = \frac{\alpha}{1-e^{\alpha T}}\int_{T}^{0}\theta(t)e^{\alpha t}dt, \tag{22}$$

and for an arbitrary time in the past,

$$p_\alpha(f_1)\Delta f = \frac{\alpha}{e^{\alpha t}-e^{\alpha T}}\int_{T}^{t}\theta(\tau)e^{\alpha\tau}d\tau. \tag{23}$$

EMP Autocorrelation and EMP Power Spectrum

The discussion here is almost identical to the concepts presented by Fano nine years ago.[5] The difference lies primarily in the fact that the concept of the emp Fourier transform is used. The result is that Wiener's theorem is reformulated exactly in the emp language, *i.e.*, the emp autocorrelation and the emp power spectrum form an exact emp Fourier pair. There is a difference in the coefficient of the emp power spectrum as compared with Fano's corresponding function. The other, rather insignificant, difference is that active analog computer circuits rather than passive networks are presented.

[5] R. M. Fano, "Short-time autocorrelation functions and power spectra," *J. Acoust. Soc. Amer.*, vol. 22, pp. 546–550; September, 1950. The writer did most of his work unaware of the work by Fano, but this paper has been modified in light of his work. Specifically, the writer originally considered the function $\psi_\alpha(\tau)$ rather than $\phi_\alpha(\tau)$ as the emp autocorrelation function. Thus, the writer considered that the emp autocorrelation function should contain the $e^{-\alpha\tau}$ term. The notation in this paper has been revised to conform almost exactly with Fano's notation. The exceptions are as follows:

a) The subscript α is used rather than the subscript t to denote the emp variables. This was retained from the author's original notation since it is anticipated that in the author's later work comparison of emp variables with different time constants, *i.e.*, α_1 and α_2, will be involved.
b) $F_\alpha(\omega)$ denotes the emp Fourier transform of $f(t)$. The use of $F_t(x)$ in Fano's sense [his (14)] is continued.

Fano's text is followed closely in various paragraphs of this section.

The emp autocorrelation function of $f(t)$ is defined as follows:

$$\phi_\alpha(\tau) = 2\alpha\int_{-\infty}^{t} f(x)f(x-\tau)e^{-2\alpha(t-x)}dx, \tag{24}$$

where τ is a positive delay. The analog computer scheme for obtaining the emp autocorrelation function is shown in Fig. 6.

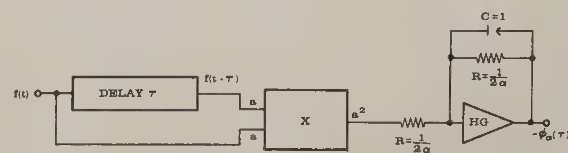

Fig. 6—Computation of $\phi_\alpha(\tau)$ in accordance with (24).

Consider now the function

$$\psi_\alpha(\tau) = 2\alpha e^{-\alpha\tau}\int_{-\infty}^{\infty} f(x)f(x-\tau)\cdot e^{-2\alpha(t-x)}u(t-x)u(t-x-\tau)dx, \tag{25}$$

where $u(t)$ is a step function described before. This function is symmetrical in τ, as can be seen by substituting $y=x-\tau$. Let the function $\phi_\alpha(\tau)$ for negative τ be interpreted as a symmetrical image of the integral in (24) taken for positive τ. Then it is possible to write:

$$\psi_\alpha(\tau) = \phi_\alpha(\tau)e^{-\alpha|\tau|}. \tag{26}$$

Note that $\psi_\alpha(\tau)$ represent a convolution integral of the function, $F_t(x)$ (here Fano's t subscript is retained):

$$F_t(x) = (2\alpha)^{1/2}f(x)e^{-\alpha(t-x)}u(t-x), \tag{27}$$

and of $F_t(-x)$. Therefore

$$\psi_\alpha(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_\alpha(\omega)g_\alpha^*(\omega)e^{j\omega\tau}d\omega, \tag{28}$$

where $g_\alpha^*(\omega)$ denotes the complex conjugate of $g_\alpha(\omega)$, and where

$$\begin{aligned} g_\alpha(\omega) &= \int_{-\infty}^{\infty} F_t(x)e^{-j\omega x}dx \\ &= (2\alpha)^{1/2}\int_{-\infty}^{\infty} f(x)e^{-\alpha(t-x)}u(t-x)e^{-j\omega x}dx \\ &= \left(\frac{2}{\alpha}\right)^{1/2}\alpha\int_{-\infty}^{t} f(x)e^{-\alpha(t-x)}e^{-j\omega x}dx. \end{aligned} \tag{29}$$

This last expression is equal to $(2/\alpha)^{1/2}$ times $F_\alpha(\omega)$, the emp Fourier transform of $f(x)$, thus

$$\begin{aligned} \psi_\alpha(\tau) &= \frac{1}{2\pi}\frac{2}{\alpha}\int_{-\infty}^{\infty} F_\alpha(\omega)F_\alpha^*(\omega)e^{j\omega\tau}d\omega \\ &= \frac{1}{2\pi}\frac{2}{\alpha}\int_{-\infty}^{\infty} P_\alpha(\omega)e^{j\omega\tau}d\omega, \end{aligned} \tag{30}$$

where $P_\alpha(\omega)$, which defines the emp power spectrum, is given by

$$P_\alpha(\omega) = F_\alpha(\omega)F_\alpha{}^*(\omega) = |F_\alpha(\omega)|^2. \tag{31}$$

Eq. (30) can be rewritten in view of (26)

$$\frac{\alpha\phi_\alpha(\tau)e^{-\alpha|\tau|}}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_\alpha(\omega)e^{j\omega\tau}d\omega. \tag{32}$$

Taking the inverse Fourier transform of (32),

$$\begin{aligned} P_\alpha(\omega) &= \frac{\alpha}{2}\int_{-\infty}^{\infty}\phi_\alpha(\tau)e^{\alpha|\tau|}e^{-j\omega\tau}d\tau \\ &= \alpha\int_{-\infty}^{0}\phi_\alpha(\tau)e^{\alpha\tau}e^{-j\omega\tau}d\tau \\ &= \text{emp Fourier transform of } \phi_\alpha(\tau). \end{aligned} \tag{33}$$

Wiener's theorem is thus restated in emp term.

Coming now to the question of experimental determination of $P_\alpha(\omega)$, it is noted that $F_\alpha(\omega)$ can be written as follows:

$$\begin{aligned} F_\alpha(\omega) &= \alpha e^{-j\omega t}\int_{-\infty}^{\infty} f(x)e^{-\alpha(t-x)}u(t-x)e^{j\omega(t-x)}dx \\ &= e^{-j\omega t}\alpha\left[\int_{-\infty}^{\infty} f(x)e^{-\alpha(t-x)}u(t-x)\cos\omega(t-x)dx \right. \\ &\quad \left. + j\int_{-\infty}^{\infty} f(x)e^{-\alpha(t-x)}u(t-x)\sin\omega(t-x)dx\right]. \end{aligned} \tag{34}$$

It follows that

$$\begin{aligned} P_\alpha(\omega) = |F_\alpha(\omega)|^2 &= \alpha^2\left[\int_{-\infty}^{t} f(x)e^{-\alpha(t-x)}\cos\omega(t-x)dx\right]^2 \\ &\quad + \alpha^2\left[\int_{-\infty}^{t} f(x)e^{-\alpha(t-x)}\sin\omega(t-x)dx\right]. \end{aligned} \tag{35}$$

The first term of the right-hand side of (35) represents the square of the output of a filter with the impulse response

$$E_\omega'(t) = \alpha e^{-\alpha t}\cos\omega t\, u(t), \tag{36}$$

and with $f(t)$ as the input. The transfer function of such filter is given by $[\alpha(\alpha+j\delta)]/[(\alpha+j\delta)^2+\omega^2]$, where δ is the frequency variable. The second term of (35) is similarly recognized as the square of the output of a filter with the impulse response

$$E_\omega''(t) = \alpha e^{-\alpha t}\sin\omega t\, u(t). \tag{37}$$

The transfer function of such filter is given by $\alpha\omega/[(\alpha+j\delta)+\omega^2]$.

The computation of $P_\alpha(\omega)$ by an analog computer setup is shown in Fig. 7.

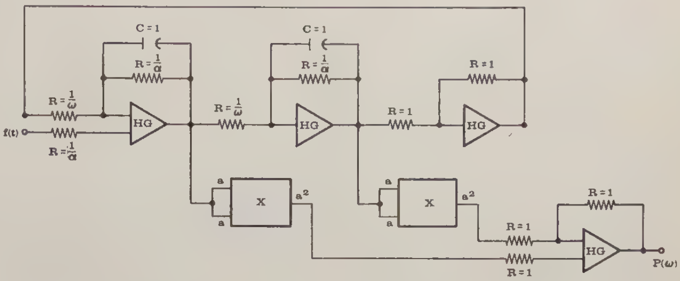

Fig. 7—Computation of $P_\alpha(\omega)$ in accordance with (35).

Discrete EMP Variables

EMP Average

The emp average of a variable $I(n)$ observed at discrete intervals Δt is defined numerically by

$$\begin{aligned} \bar{I}_q(0) &= (1-q)[I(0) + I(1)q + I(2)q^2 \cdots I(n)q^n \cdots] \\ &= (1-q)\sum_{n=0}^{N} I(n)q^n, \qquad N \to \infty, \end{aligned} \tag{38}$$

where the 0 in $\bar{I}_q(0)$ indicates that the summation extends to the most recent observation, namely $I(0)$. The factor $1-q$ is required to normalize the weighting function, since

$$\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q}, \tag{39}$$

and q^{N+1} gradually vanishes as N becomes large.

The physical meaning of $\bar{I}_q(0)$ is apparent from its numerical definition. It is an average of the observed variable I, where the recent values are weighted more heavily, and the input of a long time ago contributes insignificantly to the average. How soon the past inputs stop contributing significantly to the average depends on q. Since q generally will be very close to 1, we can say that $1-q=\epsilon$ is a small quantity, and

$$q^{1/(1-q)} = (1-\epsilon)^{1/\epsilon} \cong 1/e. \tag{40}$$

Thus, an input which bears a number $n \cong 1/(1-q)$, *i.e.*, an input that occurred approximately $\Delta t/(1-q)$ before the present, contributes by a factor e less than the most recent input. The period $\Delta t/(1-q)$ should be made smaller than the time of effective constancy of the regime. Within this restriction $1/(1-q)$ should be made large, *i.e.*, q should be as close to 1 as possible, in order to get an effective contribution to the average from a larger number of data.

The average $\bar{I}_q(0)$ will be smaller in absolute value than the "true" average [as defined later in (41)] during the initial stages of the computation, during the stages when q^{N+1} is not negligible. When it is desired to have a reasonable estimate of the average during the initial stages, the average can be defined numerically as follows:

$$\bar{I}_{q(0,N)} = \frac{1-q}{1-q^{N+1}}\left[I(0) + I(1)q + I(2)q^2 \cdots I(N)q^N\right]$$

$$= \frac{1-q}{1-q^{N+1}} \sum_{n=0}^{N} I(n)q^n. \qquad (41)$$

The flow diagram for computing the emp average in accordance with (39) is shown in Fig. 8. That corresponding to (41) is shown in Fig. 9. It will be seen that the second approach is somewhat more complicated to compute, and for the remainder of the report it will be assumed that the need for an accurate average during the initial stages does not arise.

For a moment in the past, the emp average $\bar{I}_q(n)$ is numerically defined

$$\bar{I}_q(n) = \frac{1-q}{q^n} \sum_{k=n}^{N} I(k)q^k, \qquad N \to \infty. \qquad (42)$$

This is a simple extension of $\bar{I}_q(0)$, with the difference that $\bar{I}_q(n)$ represents an average of some time ago.

EMP Variance

The emp variance $s_q^2(0)$ is defined in (43) and again in (44):

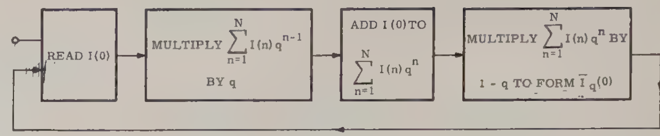

Fig. 8—Computation of $\bar{I}_q(0)$.

$$\sum_{n=0}^{N} I(n)q^n \text{ becomes } \sum_{n=1}^{N} I(n)q^{n-1},$$

once the new $I(0)$ has been read. Initially, 0 is stored in the memory cell containing

$$\sum_{n=0}^{N} I(n)q^n.$$

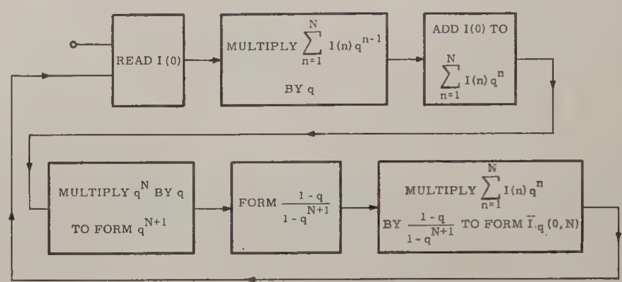

Fig. 9—Computation of $\bar{I}_q(0, N)$. q^{N+1} becomes q^N, once the new $I(0)$ has been read. Initially 1 is stored in the memory cell containing q^{N+1}.

$$s_q^2(0) = (1-q)\sum_{n=0}^{N}\left[I(n) - \bar{I}_q(n)\right]^2 q^n; \qquad (43)$$

$$\frac{s_q^2(0)}{1-q} = \sum_{n=0}^{\infty}\left[I(n) - \frac{1-q}{q^n}\sum_{k=n}^{N} I(k)q^k\right]^2 q^n$$

$$= \sum_{n=0}^{\infty}\left\{I(n) - (1-q)\left[I(n) + I(n+1)q + \cdots + I(k)q^{k-n} + \cdots\right]\right\}^2 q^n$$

$$= \sum_{n=0}^{\infty}\left\{I(n)q - (1-q)\left[I(n+1)q + I(n+2)q^2 + \cdots + I(k)q^{k-n} + \cdots\right]\right\}^2 q^n$$

$$= \left\{I(0)q - (1-q)\left[I(1)q + I(2)q + \cdots + I(n)q^n + \cdots\right]\right\}^2$$

$$+ \left\{I(1)q - (1-q)\left[I(2)q + I(3)q^2 + \cdots + I(n)q^{n-1} + \cdots\right]\right\}^2 q$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ \left\{I(n-3)q - (1-q)\left[I(n-2)q + I(n-1)q^2 + I(n)q^3 + \cdots\right]\right\}^2 q^{n-3}$$

$$+ \left\{I(n-2)q - (1-q)\left[I(n-1)q + I(n)q^2 + \cdots\right]\right\}^2 q^{n-2}$$

$$+ \left\{I(n-1)q - (1-q)\left[I(n)q + \cdots\right]\right\}^2 q^{n-1}$$

$$+ \left\{I(n)q - (1-q)\left[\cdots\right]\right\}^2 q^n$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (44)$$

TABLE I

	$-2q(1-q)I(0)q^n q^0$	$+(1-q)^2[I(n)q^{2n}+2I(n-1)q^{2n-1}\cdots 2I(1)q^{n+1}]q^0$
	$-2q(1-q)I(1)q^{n-1}q$	$+(1-q)^2[I(n)q^{2n-2}+\cdots 2I(2)q^n]q^1$
	. . . . .	. . . . . .
	$-2q(1-q)I(n-3)q^3 q^{n-3}$	$+(1-q)^2[I(n)q^6+2I(n-1)q^5+2I(n-2)q^4]q^{n-3}$
	$-2q(1-q)I(n-2)q^2 q^{n-2}$	$+(1-q)^2[I(n)q^4+2I(n-1)q^3]q^{n-2}$
	$-2q(1-q)I(n-1)qq^{n-1}$	$+(1-q)^2[I(n)q^2]q^{n-1}$
$+q^2 I(n)q^n$		

The terms with $I(n)$ that contain $I(k)$, $k \leq n$, are presented in Table I. The terms are culled from (44) above in such a way that each row in Table I corresponds to an appropriate row in (44). In the first column of Table I (which contains only one expression) is the squared term of the first term of each row of (44); in the second column of the table are the cross-terms of the first term and the terms in brackets of (44); and in the third column are the terms of multiplication inside the brackets.

Table I can be rewritten using the formula for the sum of geometric series, as follows:

$$
\begin{aligned}
I(n)q^{n+2} &- \sum_{k=1}^{n-1} 2q^{n+1}(1-q)I(k) - 2q^{n+1}(1-q)I(0) \\
&+ (1-q)^2 I(n)(q^{n+1}+q^{n+2}+\cdots+q^{2n}) \\
&+ 2(1-q)^2 \sum_{k=1}^{n-1} I(k)(q^{n+1}+q^{n+2}\cdots q^{n+k}) \\
= I(n)q^{n+2} &- \sum_{k=1}^{n-1} 2q^{n+1}(1-q)I(k) \\
&+ (1-q)^2 I(n)q^{n+1}\frac{1-q^n}{1-q} \\
&- 2q^{n+1}(1-q)I(0) + 2(1-q)^2 \sum_{k=1}^{n-1} I(k)q^{n+1}\frac{1-q^k}{1-q} \\
= q^{n+1}\Big[I(n)q &+ I(n) - I(n)q - I(n)q^n + I(n)q^{n+1} \\
&- 2(1-q)I(0) - 2(1-q)\sum_{k=1}^{n-1} I(k)q^k\Big]. \qquad (45)
\end{aligned}
$$

Theorem: Let $\overline{I_q^2}$ be defined as follows:

$$\overline{I_q^2}(0) = (1-q)\sum_{n=0}^{\infty} I^2(n)q^n. \qquad (46)$$

The following relationship will now be demonstrated:

$$q s_q^2(0) = \overline{I_q^2}(0) - \overline{I}_q^2(0). \qquad (47)$$

Consider the expression $\overline{I_q^2}(0) - \overline{I}_q^2(0)$:

$$
\begin{aligned}
&\overline{I_q^2}(0) - \overline{I}_q^2(0) \\
&= (1-q)\sum_{n=0}^{\infty} I^2(n)q^n - (1-q)^2\left[\sum_{n=0}^{\infty} I(n)q^n\right]^2; \qquad (48)
\end{aligned}
$$

and, after dividing by $(1-q)$, expand as follows:

$$
\begin{aligned}
\frac{\overline{I_q^2}(0) - \overline{I}_q^2(0)}{1-q} &= I^2(0) + I^2(1)q + I^2(2)q^2 + \cdots \\
&+ I^2(n)q^n + \cdots - (1-q)[I(0) + I(1)q \\
&+ I(2)q^2 + \cdots I(n)q^n + \cdots]^2. \qquad (49)
\end{aligned}
$$

The terms with $I(n)$ which contain only $I(k)$, such that $k \leq n$, are

$$
\begin{aligned}
I(n)q^n &- 2(1-q)[I(0) + I(1)q + \cdots + I(n-1)q^{n-1}]q^n \\
&- (1-q)I(n)q^{2n} = q^n[I(n) - I(n)q^n + I(n)q^{n+1} \\
&- 2(1-q)I(0) - 2(1-q)\sum_{k=1}^{n-1} I(k)q^k\Big]. \qquad (50)
\end{aligned}
$$

A comparison of terms of (50) with terms given in (45) establishes the validity of (47).

Conclusion

The exponentially-mapped-past statistical variables offer considerable advantages in either digital or analog computation. The analog computer circuits are very simple and easy to realize. Neither perfect integrators nor delay devices are required, elements which are necessary for instance if statistical analysis is based on an equally-weighted finite aperture extending from the present to time T in the past. The digital computer programs involve storing only one or several emp variables, rather than storing all the data for the interval of interest.

The theoretical relationships which have been presented are simple and show parallelism with ordinary statistical variables.

While this report does not present a complete theory of exponentially-mapped-past statistical variables, it attempts to present a unified approach to the emp statistics.

Acknowledgment

The writer wishes once more to call the reader's attention to the pioneering work of R. M. Fano. The fact that his paper[5] was published in a journal that is not bread-and-butter reading of statisticians and control engineers was amply demonstrated to the writer; several statisticians and control engineers with whom the writer discussed the emp variables, and an even larger audience before which some of this work was originally presented, were unacquainted with Fano's work. This is highly regrettable.

Generalized Weighting Function and Restricted Stability of a Linear Pulse-Modulated Error Feedback System*

WILLIAM A. JANOS†

Summary—**A generalized weighting function is obtained for a linear feedback system with a pulse-modulated error signal. This is expressed in the form of a matrix operator acting on an input vector, the components of which are the first $R-1$ derivatives of the input, where R is the order of the unmodulated closed-loop system. In addition, since the system operator takes the form of a finite dimensional matrix, it has been possible to make and realize more stringent conditions on the transient stability; namely, a preassigned bounded output after a preassigned time.**

Although the solution has been obtained generally for nonuniform pulsing, the latter stability investigation has been made only for the uniform case.

THE present article is concerned with the inversion of the integral equation relating output to inputed in a linear feedback system with a pulse-modulated error signal. Both uniform and nonuniform pulsing are considered. The formal solution is expressible in closed form as a matrix operator of rank equal to the order of the closed-loop system, acting on an appropriately defined input vector.

Previous work on this problem has been carried out primarily by Farmanfarma.[1] In such an investigation the analysis was performed in the transform (s) domain, to yield a comprehensive treatment of the problem in terms of the P-transform. Stability studies of an asymptotic nature were also included in the latter work.

Although the Farmanfarma study is most commendable and penetrating, it seems perhaps even formally unwieldy. The present treatment essentially obtains a generalized weighting function for the system in the form of a matrix operator acting on an input vector, the components of which are the first $R=1$ derivatives of the input, where R is the order of the unmodulated closed-loop system. Thus, the present approach may have certain pictorial merit. Although it is formally more concise, the calculational work necessary for an arbitrarily ordered system can become rather involved too. In addition, since the system takes the form of a finite dimensional matrix, it has been possible to make and realize more stringent conditions on the transient stability; namely, a preassigned bounded output after a preassigned time. Although the solution has been obtained generally for nonuniform pulsing, the latter stability investigation has been made only for the uniform case.

The following system is given (see Fig. 1) where G_1

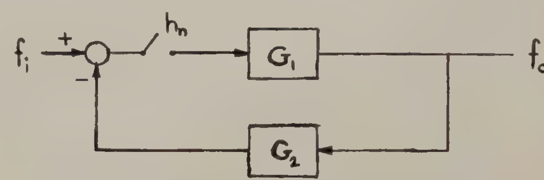

Fig. 1—System block diagram.

and G_2 label the components with corresponding impulse responses $G_1(\tau)$ and $G_2(\tau)$, respectively. The pulse modulation is represented by the crossed switch of duration h_n over the nth interval. The integral equation governing this system has the form,

$$\int_0^t G_1(t-\tau)u(\tau)\left\{f_i(\tau) - \int_0^\tau G_2(\tau-\tau')f_0(\tau')d\tau'\right\}d\tau = f_0(t) \qquad (1)$$

where

$$\begin{aligned} u(\tau) &= 1, \quad T_n \leq \tau \leq T_n + h_n, \quad \text{all integers } n, \\ &= 0, \quad \text{otherwise}; \end{aligned} \qquad (2)$$

f_i is assumed turned on at $\tau=0$. Let

$$\begin{aligned} &\frac{N_1(p)}{D_1(p)} \text{ be the Laplace transform of } G_1(\tau), \\ &\frac{N_2(p)}{D_2(p)} \text{ the transform of } G_2(\tau). \end{aligned} \qquad (3)$$

Further, let us make the following definitions, for all functions f concerned:

$$\begin{aligned} f_n(\tau) &= f(\tau), \quad T_n \leq \tau \leq T_n + h_n. \\ &= 0, \quad \text{otherwise}. \\ \bar{f}_n(\tau) &= f(\tau), \quad T_{n-1} + h_{n-1} \leq \tau \leq T_n \\ &= 0, \quad \text{otherwise}. \end{aligned} \qquad (4)$$

Note that the transforms of $f_n, \bar{f}_n$ would be the P-transforms of f as expressed previously.[1]

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[1] G. Farmanfarma, "General Analysis of Finite Pulsed Linear Systems," Ph.D. dissertation, University of California, Berkeley; November, 1957.

Hence, over the interval (T_n, T_n+h_n) we have the following operational form of (1).

$$f_{0n}(T_n+\tau) = \frac{N_1(p)}{D_1(p)}\left\{f_{in}(T_n+\tau') - \frac{N_2(p)}{D_2(p)} f_{0n}(T_n+\tau'')\right\}$$
$$0 \le \tau, \tau', \tau'' \le h_n \quad (5)$$

where the integral operators,

$$\frac{N_1(p)}{D_1(p)} \quad \text{and} \quad \frac{N_1(p)}{D_1(p)} \cdot \frac{N_2(p)}{D_2(p)}$$

refer to the variables τ', τ''.

After transposing and integrating by parts, (5) takes the form

$$\begin{aligned} f_{0n}(T_n+\tau) = {} & \{N_1(p)N_2(p) + D_1(p)D_2(p)\}^{-1} \\ & \cdot \{D_2(p)N_1(p)f_{in}(T_n+\tau') \\ & + \sum_{m=1}^{R} \gamma_m \sum_{r=0}^{m-1} p^{m-1-r} \bar{f}_{0n}^{(r)}(r)(T_n)\} \end{aligned} \quad (6)$$

where

$$f_{0n}^{(r)}(T_n) \equiv \left(\frac{d^r}{d\tau^r}\bar{f}_{0n}(\tau)\right)_{\tau=T_n},$$

and

$$N_1(p)N_2(p) + D_1(p)D_2(p) \equiv \sum_{m=0}^{R} \gamma_m p^m; \quad (7)$$

i.e., γ_m is the coefficient of p^m in the polynomial in p, $N_1N_2+D_1D_2$. Notice that the last term on the right of (6) represents the influence of the initial conditions at $\tau=T_n+$, which are the terminal conditions of $\bar{f}_{0n}$. However, during the preceding interval, $(T_{n-1}+h_{n-1}, T_n)$, we have

$$\begin{aligned} & \bar{f}_{0n}(T_{n-1}+h_{n-1}+\tau'') \\ & = \frac{1}{D_1(p)} \sum_{m=1}^{Q} \beta_m \left\{\sum_{r=0}^{m-1} p^{m-1-r} f_{0n-1}^{(r)}(T_{n-1}+h_{n-1})\right\} \end{aligned}$$
$$0 \le \tau'' \le T_n - T_{n-1} - h_{n-1} \quad (8)$$

where

$$D_1(p) = \sum_{m=0}^{V} \beta_m p^m. \quad (9)$$

Since there is no driving function or feedback over this interval, the response is the decay characteristic of G_1 with the initial conditions being, again, the terminal conditions of the response over the previous interval, f_{0n-1}. Thus, we can establish recursion relations over consecutive intervals.

Let us first give interpretation to the operational symbolism used. Consider the following, where the van der Pol-Bremmer transform pair notation is used.[2]

[2] B. van der Pol and H. Bremmer, "Operational Calculus Based on the Two-Sided Laplace Integral," Cambridge University Press, Cambridge, Eng., 2nd ed.; 1955.

$$\frac{1}{D_1(p)} \doteqdot \sum_{j=1}^{V} w_{1j}(\tau), \quad (10)$$

$$\frac{1}{N_1(p)N_2(p) + D_1(p)D_2(p)} \doteqdot \sum_{k=1}^{R} w_{2k}(\tau), \quad (11)$$

$$\frac{D_2(p)N_1(p)F_{in}(T_n+\tau'')}{N_1(p)N_2(p) + D_1(p)D_2(p)} = \mathfrak{F}_i(T_n+\tau''). \quad (12)$$

On using (10) in (8) and evaluating at $\tau''=T_n-T_{n-1}-h_{n-1}$, it is useful to define the following matrices; here $\{A\}_{ij}$ denotes the i, jth element of the matrix A.

$$\{\mathcal{W}_1(\tau)\}_{qr} = \sum_{j=1}^{V} \sum_{m=1+r'}^{V} \beta_m w_{ij}^{(q+m-1-r)}(\tau), \qquad m \ge 1+r$$
$$(R \times V), \quad r = 0, 1, \cdots, V-1 \quad (13)$$

$$\{\mathcal{W}_2(\tau)\}_{q'r'} = \sum_{k=1}^{R} \sum_{m=1}^{R} \gamma_m w_{2K}^{(q'+m-1-r')}(\tau), \qquad m \ge 1+r',$$
$$(V \times R), \quad r' = 0, 1, \cdots, R-1 \quad (14)$$

$$\overset{T}{F}_{0n-1}(\tau) = \{f_{0n-1}(T_{n-1}+\tau), f_{0n-1}^{(1)}(T_{n-1}+\tau), \cdots,$$
$$f_{0n-1}^{(V-1)}(T_{n-1}+\tau)\}, \quad (V \times 1) \quad (15)$$

$$\overset{T}{\phi}_0 = \{f_{00}(0), f_{00}^{(1)}(0), \cdots, f_{00}^{(R-1)}(0)\}, (R \times 1) \quad (16)$$

$$\overset{T}{F}_{in}(\tau) = \{\mathfrak{F}_{in}(T_n+\tau), \mathfrak{F}_{in}^{(1)}(T_n+\tau), \cdots,$$
$$\mathfrak{F}_{in}^{(V-1)}(T_n+\tau)\}, \quad (V \times 1). \quad (17)$$

The T superscript denotes the transpose, and "$(x\times y)$" denotes x rows and y columns. Further, it is clear that

$$f_{0n}^{(p)}(T_n) = \bar{f}_{0n}^{(p)}(T_n), \qquad p = 0, 1, \cdots, V-1; \quad (18)$$

namely, the initial conditions of the first $V-1$ derivatives of $f_{0n}(T_n+\tau)$ are the terminal conditions of $\bar{f}_{0n}(T_{n-1}+h_{n-1}+\tau)$. Thus, on using matrices (13) through (17) and (18), the first $V-1$ derivatives of (6), evaluated at the terminal points of each interval, may be expressed by the matrix equation

$$\begin{aligned} F_{0n}(h_n) = {} & \mathcal{W}_2(h_n)\mathcal{W}_1(T_n - T_{n-1} - h_{n-1})F_{0n-1}(h_{n-1}) \\ & + F_{in}(h_n). \end{aligned} \quad (19)$$

Eq. (19) is a recurrence relation between $F_{0n}(h_n)$ and $F_{0n}(h_{n-1})$. Hence, we may recur back to ϕ_0, (16), and obtain

$$F_{0n}(h_n) = \prod_{n'=1}^{n} \{\mathcal{W}_2(h_{n'})\mathcal{W}_1(\Delta T_{n'} - h_{n'-1})\}\mathcal{W}_2(h_0)\phi_0$$

$$+ \sum_{n'=0}^{n} \prod_{n''+1}^{n} \{\mathcal{W}_2(h_{n''})\mathcal{W}_1(\Delta T_{n''} - h_{n''-1})\}F_{in'}(h_{n'}) \quad (20)$$

where

$$\Delta T_{n'} = T_{n'} - T_{n'-1} \quad (21)$$

and $\prod$ denotes the ordered product; *i.e.*, for $n \geq n'$, matrices G_K,

$$\prod_{n'}^{n} G_K \equiv G_n \cdot G_{n-1} \cdots G_{n'},$$

$\prod_{n+1}^{n}$ is defined to be unity in the following sense:

$$\prod_{n+1}^{n} G_K \equiv G_n. \tag{22}$$

The first term on the right of (20) denotes the response due to initial conditions at $t=0$, the second denotes that due to the driving function f_i.

Thus, the response at the terminal point of each pulsing interval, $T_n + h_n$, has been obtained. The output behavior during the interval is given by a substitution of (20) into

$$F_{0n}(\tau) = -(\mathcal{W}_2(\tau)\mathcal{W}_1(\Delta T_n - h_{n-1})F_{0n-1}(h_n) + F_{i_n}(\tau) \quad 0 \leq \tau \leq h_n. \tag{23}$$

The response between pulsing intervals is given by (8), with (10) and (20) substituted.

Uniform Pulsing

The general solution (20) takes on a less complicated form for the uniformly pulsed case. Here

$$T_n = nT, \qquad h_n = h. \tag{24}$$

Then we have

$$F_{0n}(h) = \left\{\mathcal{W}_2(h)\mathcal{W}_1(T-h)\right\}^n \mathcal{W}_2(h)\phi_0 + \sum_{n'=0}^{n} \left\{\mathcal{W}_2(h)\mathcal{W}_1(T-h)\right\}^{n-n'} F_{i_{n'}}(h). \tag{25}$$

or, to express the weighting function for the response at the end of each pulse interval,

$$F_{0n}(h) = \sum_{n=0}^{m} \mathcal{W}_3^{\,m-n}\left\{F_{in}(h) + \delta_{n0}W_2(h)\phi_0\right\}$$

where

$$\mathcal{W}_3 = \mathcal{W}_2(h)\mathcal{W}_1(T-h) \tag{26}$$

and

$$\delta_{n0} = 1, \qquad n = 0$$
$$= 0, \qquad \text{otherwise.}$$

Hence, $\mathcal{W}_3^m$ is the generalized weighting function at the pulse terminal points for the uniformly pulsed case. Eq. (26) could have been obtained also by substituting (24) into (19) and using the z-transform representation.[3] A substitution of (24) and (26) into (23) and (8) then determines the continuous response during and between pulsing intervals, respectively.

It is seen that in the solution (26) the matrix $\mathcal{W}_2(h)\mathcal{W}_1(T-h)$ must be raised to integral powers. Considering (13) and (14), we see that this matrix must be square, $V \times V$. Assuming that it is of full rank V of distinct eigenvalues (of zero multiplicity), then $\{W_2(h)W_1(T-h)\}^m$, $m > V$, is reducible by the Cayley-Hamilton theorem,[4] to a polynomial of degree not greater than $V-1$. This can be stated in another way. Let

$$\mathcal{W}_2(h)\mathcal{W}_1(T-h) = S^{-1}\Lambda S,$$

where Λ is the matrix of eigenvalues of $\mathcal{W}_2(h)\mathcal{W}_1(T-h)$ and S is the generator of the similarity transformation which diagonalizes it.

Then

$$\left\{\mathcal{W}_2(h)\mathcal{W}_1(T-h)\right\}^m = S^{-1}\Lambda^m S.$$

Therefore, in order to raise such a matrix to a power, the eigenvalues and matrix S are necessary.[4]

Stability Considerations for Uniform Pulsing

Let us require the following behavior on the transient response. Given (N, ϵ_N) such that

$$\|F_{0n}\| \equiv \max_{\tau} |F_{0n}(h)| \leq \epsilon_N, \quad \text{for all } n \geq N \tag{27}$$

where

$$\max_{\tau} |F_{0n}(h)|$$

denotes the largest member of the R-tuple:

$$(|f_{0n}(h)|, |f_{0n}^{(1)}(h)|, \cdots, |f_{0n}^{(r)}(h)|, \cdots, |f_{0n-1}^{(R-1)}(h)|).$$

Let us define the following norm for matrices:

$$\|A\| \equiv \max_{i} \sum_{j} |\{A\}_{ij}| \equiv \sup_{x} \frac{\|Ax\|}{\|x\|}, \tag{28}$$

where x is a column matrix; *i.e.*, the norm is the maximal row vector. It is evident that the norm in the case of a column matrix reduces to that of (27). The consequence of the following derivation, from functional analysis,[5] is to be found very useful. Given matrices A, B and column vector x. Then from (27), (28)

$$\|BAx\| \leq \|BA\| \cdot \|x\| \leq \|B\| \cdot \|A\| \cdot \|x\|,$$

or equivalently

$$\|BA\| \leq \|B\| \cdot \|A\|. \tag{29}$$

[3] E. I. Jury, "Analysis and synthesis of sampled data control systems," *Trans. AIEE*, vol. 73, pt. 1, pp. 332–346; September, 1954.

[4] B. Friedman, "Principles and Techniques of Applied Mathematics, J. Wiley and Sons, Inc., New York, N. Y., 1956.

[5] A. N. Kolmogorov and S. V. Fomin, "Functional Analysis: Metric and Normed Spaces," Graylock Press, Rochester, N. Y., vol. 1; 1959.

It is assumed that we are free to modify the feedback component G_2 in order to attempt to fulfill (27). Since the response to an initial condition vector ϕ_{01} (16), is explicitly a function of $\mathcal{W}_2(h)$, (26), which contains the parameters of G_2 implicitly, [(7) and (17)], the problem is first to obtain bounds on $\mathcal{W}_2$ satisfying (27), and then to derive suitable parameters for G_2. Thus, from (27)

$$F_{0n} = (\mathcal{W}_2\mathcal{W}_1)^N\mathcal{W}_2\phi_0. \tag{30}$$

Then, according to definition (28) and result (29)

$$\|F_{0n}\| = \|(\mathcal{W}_2\mathcal{W}_1)^N\mathcal{W}_2\phi_0\| \leq \|\mathcal{W}_2\|^{N+1}\cdot\|\mathcal{W}_1\|^N\cdot\|\phi_0\|. \tag{31}$$

Thus, if

$$\|\mathcal{W}_2\| \leq \left(\frac{\epsilon_N}{\|\mathcal{W}_1\|^N\|\phi_0\|}\right)^{1/(N+1)} \tag{32}$$

it follows that

$$\|F_{0N}\| \leq \epsilon_N. \tag{33}$$

But now we must also have boundedness for all $n>N$. Consider the original recurrence relation with zero "driving" function.

$$F_{0_{n+1}} = \mathcal{W}_2(h)\mathcal{W}_1(T-h)F_{0_n}. \tag{34}$$

Thus

$$\|F_{0_{N+1}}\| \leq \|\mathcal{W}_2(h)\|\|\mathcal{W}_1(T-h)\|\|F_{0_N}\|. \tag{35}$$

Hence, if

$$\|\mathcal{W}_2(h)\| \leq \min\left(\left(\frac{\epsilon_N}{\|\mathcal{W}_1\|N\|\phi_0\|}\right)^{1/(N+1)}, \frac{1}{\|\mathcal{W}_1\|}\right) \tag{36}$$

then (32) will be fulfilled and also

$$\|F_{0_{n+1}}\| \leq \|F_{0_n}\| \qquad \text{for all } n \geq N, \tag{37}$$

all of which imply

$$\|F_{0_n}\| \leq \epsilon_N \qquad \text{for all } n \geq N.$$

It also follows that if only the inequality in (36) is fulfilled, the vector sequence $\{F_{0_n}\}$ converges to zero faster than some geometric sequence,

$$\{\phi^m\}, \qquad |\phi| < 1.$$

Example A

In the example we shall consider the response to a unit step of a second order system in which g_1 is of second order and g_2 is a gain constant. Although calculational details will be carried out to an extent, they are to be secondary to the derivation of the general form of the solution.

Let us first consider the general form of 2×2 matrix raised to a power

$$\mathcal{W}_3 = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}. \tag{38}$$

The eigenvalues of Ω are determined by

$$\text{det.}\,(\mathcal{W}_3 - \lambda I) = \begin{vmatrix} w_{11}-\lambda & w_{12} \\ w_{21} & w_{22}-\lambda \end{vmatrix} = 0. \tag{39}$$

Thus

$$\lambda_1 = \frac{w_{11}+w_2}{2} + \sqrt{\frac{(w_{11}+w_{22})^2}{4} - w_{11}w_{22} + w_{12}w_{21}},$$

$$\lambda_2 = \lambda_1 - 2\sqrt{\frac{(w_{11}+w_{12})^2}{4} - w_{11}w_{22} + w_{12}w_{21}}. \tag{40}$$

Now, to determine the generator S of the similarity transformation,

$$S\colon S\Omega S^{-1} = \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{41}$$

and

$$\Omega^m = S^{-1}\Lambda^m S. \tag{42}$$

So the following must be solved:

$$S\Omega = \Lambda S \tag{43}$$

or

$$\Omega\begin{pmatrix} S_{11} \\ S_{12} \end{pmatrix} = \begin{pmatrix} \lambda_1 & S_{11} \\ \lambda_1 & S_{12} \end{pmatrix}$$

and

$$\Omega\begin{pmatrix} S_{21} \\ S_{22} \end{pmatrix} = \begin{pmatrix} \lambda_2 & S_{21} \\ \lambda_2 & S_{22} \end{pmatrix} \tag{44}$$

which yields as a possible solution

$$S_{11} = \frac{-w_{21}S_{12}}{w_{11}-\lambda_1}, \qquad S_{21} = \frac{-w_{21}S_{22}}{w_{11}-\lambda_2}. \tag{45}$$

Therefore, we may set $S_{12}=S_{22}=1$ and obtain

$$S_{11} = \frac{-w_{21}}{w_{11}-\lambda_1}, \qquad S_{21} = \frac{-w_{21}}{w_{11}-\lambda_2}. \tag{46}$$

The inverse of S, $S^{-1}\colon SS^{-1}=S^{-1}S=I$

$$S^{-1} \equiv \begin{pmatrix} \sigma_{12} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

$$\sigma_{11} = \frac{1}{w_{12}}\,\frac{(w_{11}-\lambda_1)(w_{11}-\lambda_2)}{\lambda_2-\lambda_1} = -\sigma_{12}$$

$$\sigma_{21} = -\frac{(w_{11}-\lambda_2)}{\lambda_2-\lambda_1} = -\sigma_{22}. \tag{47}$$

Thus

$$\Omega^m = S^{-1}\Lambda^m S$$

$$= \begin{pmatrix} \sigma_{11}\lambda_1{}^mS_{11}+\sigma_{12}\lambda_2{}^mS_{21}, & \sigma_{11}\lambda_1{}^mS_{12}+\sigma_{12}\lambda_2{}^mS_{21} \\ \sigma_{21}\lambda_1{}^mS_{11}+\sigma_{22}\lambda_2{}^mS_{21}, & \sigma_{21}\lambda_1{}^mS_{12}+\sigma_{22}\lambda_2{}^mS_{22} \end{pmatrix}. \tag{48}$$

We shall keep this result for later reference.

As already mentioned, G_1 is a second order system and G_2 is a constant gain factor. Hence

$$N_1 = Ap + B, \tag{49}$$

$$D_1 = (p+\alpha_1)(p+\alpha_2) = p^2 + (\alpha_1+\alpha_2)p + \alpha_1\alpha_2$$

$$= \sum_{m=0}^{2} \beta_m p^m \tag{50}$$

$$\frac{1}{D_1} \doteq \frac{1}{\alpha_2 - \alpha_1}\left\{e^{-\alpha_1 t} - e^{-\alpha_2 t}\right\} = v_1(t) \tag{51}$$

$$N_2 = 1, \qquad D_2 = K, \qquad G_2 = \frac{1}{K}, \tag{52}$$

$$N_1N_2 + D_1D_2$$

$$= K\left(p_2 + \left((\alpha_1+\alpha_2) + \frac{A}{K}\right)p + \alpha_1\alpha_2 + \frac{B}{K}\right)$$

$$= \sum_{m=0}^{2} \gamma_m p^m \tag{53}$$

$$\frac{1}{N_1N_2 + D_1D_2}$$

$$= \left(2K\sqrt{\frac{1}{4}\left((\alpha_1+\alpha_2) + \frac{A}{K}\right)^2 - \left(\alpha_1\alpha_2 + \frac{B}{K}\right)}\right)^{-1}$$

$$\cdot\left\{e^{-(\alpha_1+\alpha_2+A/K+\sqrt{\ })t} - e^{-(\alpha_1+\alpha_2+A/K-\sqrt{\ })t}\right.$$

$$= v_2(t) = K_2(e^{-C_1 t} - e^{-C_2 t}). \tag{54}$$

Thus, from (13) and (17)

$$\mathfrak{W}_1(t) = \begin{pmatrix} \beta_1 v_1(t) + \beta_2 v_1'(t), & \beta_2 v_1(t) \\ \beta_1 v_1'(t) + \beta_2 v_1''(t), & \beta_1 v_1'(t) + \beta_2 v_1'(t) \end{pmatrix}, \tag{55}$$

$$\mathfrak{W}_2(t) = \begin{pmatrix} \gamma_1 v_2(t) + \gamma_2 v_2'(t), & \gamma_2 v_2(t) \\ \gamma_1 v_2(t) + \gamma_2 v_2(t), & \gamma_1 v_2(t) + \gamma_2 v_2'(t) \end{pmatrix}. \tag{56}$$

Let the input f_i be a unit step

$$f_i(\tau) = H(\tau)\frac{1}{p} \tag{57}$$

$$f_{i_n}(nt+\tau) = H(t-nT) - H(t-nT+h). \tag{58}$$

Thus, from (12)

$$F_i(nt+\tau) = \frac{D_2N_1}{N_1N_2 + D_1D_2} f_{in}(nT+\tau) \tag{59}$$

and from (49) and (54),

$$\frac{D_2N_1}{N_1N_2+D_1D_2} = \left(KA\frac{d}{d\tau} + B\right)v_2(\tau) \tag{60}$$

$$\therefore\ \mathfrak{F}_i(nt+\tau) = \int_{nT}^{nt+\tau}\left(KA\frac{d}{d\tau} + B\right)v_2(\tau)d\tau.$$

$$\mathfrak{F}(nT+\tau) = KAK_2\{(e^{-C_1\tau}-1)e^{-C_1nt} + (e^{-C_2\tau}-1)e^{-C_2nT}\}$$

$$- BK_2\left\{\frac{1}{C_1}(e^{-C_1\tau}-1)e^{-C_1nt}\right.$$

$$\left. + \frac{1}{C_2}(e^{-C_2\tau}-1)e^{-C_2nt}\right\}, \tag{61}$$

or

$$\mathfrak{F}_i(nT+\tau) = \underline{f_{00}e^{-C_1nT} + f_{01}e^{-C_2nT}},$$

$$\mathfrak{F}_i'(nT+\tau) = \underline{f_{10}e^{-C_1nT} + f_{11}e^{-C_2nT}}. \tag{62}$$

Zero initial conditions are to be assumed for this case. Then (26) becomes

$$F_{0n}(h) = \sum_{n=0}^{m} \mathfrak{W}_3^{m-n}F_{in}(h),$$

where

$$\mathfrak{W}_3 = \mathfrak{W}_2(h)\mathfrak{W}_1(T-h). \tag{63}$$

Hence the output and first derivative of the output take the form

$$f_{0n}(h) = \sum_{n=0}^{m}\left\{\{\mathfrak{W}_3^{m-n}\}_{11}F_1(nT+h)\right.$$

$$\left. + \{\mathfrak{W}_3^{m-n}\}_{12}F_1'(nT+h)\right\}$$

$$f_{0n}'(h) = \sum_{n=0}^{m}\left\{\{\mathfrak{W}_3^{m-n}\}_{21}F_1(nT+h)\right.$$

$$\left. + \{\mathfrak{W}_3^{m-n}\}_{22}F_1'(nT+h)\right\}. \tag{64}$$

Disregarding calculational details, usage of (48), and (62) yield the following form of the output vector $F_{0n}(h)\cdots$:

$$f_{0n}(h) = \sum_{0}^{m}\left\{\lambda_1^{m-n}\{\zeta_{11}^0e^{-C_1nT} + \zeta_{12}^0e^{-C_2nT},\right.$$

$$\left. + \lambda_2^{m-n}\{\zeta_{21}^0e^{-C_1nT} + \zeta_{22}^0e^{-C_2nT}\}\right\}$$

$$f_{0n}'(h) = \sum_{0}^{m}\left\{\lambda_1^{m-n}\{\zeta_{11}'e^{-C_1nT} + \zeta_{12}'e^{-C_2nT}\right.$$

$$+ \lambda_2^{m-2}\{\zeta_{21}'e^{-C_1nT} + \zeta_{22}'e^{-C_2nT}\}; \tag{65}$$

where λ_1, λ_2 are the eigenvalues of $\mathfrak{W}_3 = (\mathfrak{W}_2(h)\mathfrak{W}_1(t-h))$, $(A_1) - (A_3)$, and the ζ_{ij}^0, ζ_{ij}' are obtained from (48) and (62) as products of elements of the S, S^{-1} matrices (41) and (47) and the coefficient f_{ij} of (62). Hence, the vector $F_{0n}(h)$ is available in closed form since the geometric progressions indicated are easily summable.

$$f_{0m}{}^{\gamma}(h) = \sum_{i,j=1}^{2} \zeta_{ij}{}^{\gamma} \frac{\lambda_i{}^m + \lambda_1{}^{-1}}{1-\lambda_1{}^{-1}e^{-C_{it}}}, \quad \gamma = 0, 1. \tag{66}$$

$$F_{0m}(h) = \begin{bmatrix} f_{0m}(h) \\ \\ f_{0m}'(h) \end{bmatrix} = \begin{bmatrix} \zeta_{11} \dfrac{\lambda_1{}^m - \lambda_1{}^{-1}e^{-C_1mT}}{1-\lambda_1{}^{-1}e^{-C_1T}} + \zeta_{12}' \dfrac{\lambda_1{}^m - \lambda_1{}^{-1}e^{-C_2mT}}{1-\lambda_1{}^{-1}e^{-C_2T}} + \zeta_{21} \dfrac{\lambda_2{}^m - \lambda_2{}^{-1}e^{-C_1mT}}{1-\lambda_2{}^{-1}e^{-C_1T}} + \zeta_{22} \dfrac{\lambda_2{}^m - \lambda_2{}^{-1}e^{-C_2mT}}{1-\lambda_2{}^{-1}e^{-C_2T}} \\ \zeta_{11}' \dfrac{\lambda_1{}^m - \lambda_1{}^{-1}e^{-C_1mT}}{1-\lambda_1{}^{-1}e^{-C_1T}} + \zeta_{12}' \dfrac{\lambda_1{}^m - \lambda_1{}^{-1}e^{-C_2mT}}{1-\lambda_1{}^{-1}e^{-C_2T}} + \zeta_{21} \dfrac{\lambda_2{}^m - \lambda_2{}^{-1}e^{-C_1mT}}{1-\lambda_2{}^{-1}e^{-C_1T}} + \zeta_{22}' \dfrac{\lambda_2{}^m - \lambda_2{}^{-1}e^{-C_2mT}}{1-\lambda_2{}^{-1}e^{-C_2mT}} \end{bmatrix}.$$

Example B

Here, the system described in Example A will be stabilized. It will be assumed that the forward loop component G_1 is inalterable; hence, we must obtain a proper bounding value for $G_2=1/K$ in order to have the transient response behave according to prescription. Given the initial condition vector,

$$\phi_0 = \begin{bmatrix} f_{00} \\ f_{00}' \end{bmatrix},$$

it is required that the transient response to ϕ_0 decay to ϵ_N or less on and after an output time NT. Thus, according to (35)

$$\|\mathcal{W}_2(h)\| \le \min\left(\left(\frac{\epsilon_N}{\|\mathcal{W}_1(t-h)\|^N\|\phi_0\|}\right)^{1/N+1}, \frac{1}{\|\mathcal{W}_1(t-h)\|}\right).$$

The following numerical values for the parameters concerned are to be used . . .

$$\begin{aligned} f_{00} &= 0, & \epsilon_N &= 0.001, \\ f_{00}' &= 1, & N &= 10, \\ A &= 2, & T &= 10, \\ B &= 1, & h &= 3. \\ \alpha_1 &= -0.5, \\ \alpha_2 &= 1, \end{aligned} \tag{67}$$

Thus, from (51) and (67)

$$v_1(t) = \tfrac{2}{3}\{e^{0.5t} - e^{-t}\}, \tag{68}$$

and from (50)

$$\beta_1 = (\alpha_1 + \alpha_2) = \tfrac{1}{2}, \qquad \beta_2 = 1, \tag{69}$$

and on referring to (55) and where $T-h=7$,

$$\mathcal{W}_1(T-h) = \begin{pmatrix} \frac{2}{3}e^{0.35} + \frac{1}{3}e^{-0.7}, & \frac{2}{3}(e^{0.35} - e^{-0.7}) \\ \frac{1}{3}(e^{0.35} - e^{-0.7}), & \frac{2}{3}e^{0.35} + \frac{1}{3}(e^{0.7}) \end{pmatrix} = \begin{pmatrix} 0.45, & 0.62 \\ 0.93, & 0.45 \end{pmatrix}. \tag{70}$$

It is seen that the norm of $\mathcal{W}_1(T-h)$, as defined by (28), is

$$\|\mathcal{W}_1(T-h)\| = 0.93 + 0.45 = 1.38. \tag{71}$$

Further, from (67),

$$\phi_0 = \max(0, 1) = 1. \tag{72}$$

Hence,

$$\|\mathcal{W}_2(h)\| \le \min\left(\left(\frac{\epsilon_N}{\|\mathcal{W}_1(T-h)\|^N\|\phi_0\|}\right)^{1/N+1}, \frac{1}{\|\mathcal{W}(T-h)\|}\right) = \min\left(\frac{1}{1.38}, \frac{1}{1.38}(0.00138)^{1/11}\right). \tag{73}$$

Now $(0.00138)^{1/11}=0.55$. Thus

$$\|\mathcal{W}_2(h)\| \le 0.39. \tag{74}$$

Referring to (54) and (67),

$$K_2 = \left[2K\sqrt{\frac{\left(\alpha_1+\alpha_2+\frac{A}{K}\right)^2}{4} - \left(\alpha_1\alpha_2 + \frac{B}{K}\right)}\right]^{-1} = \left(\frac{9K^2}{4} - 2K + 1\right)^{-1/2}, \tag{75}$$

$$C_1 = \alpha_1 + \alpha_2 + \frac{A}{K} + \sqrt{\frac{\left(\alpha_1+\alpha_2+\frac{A}{K}\right)^2}{4} - \left(\alpha_1\alpha_2 + \frac{B}{K}\right)} = \frac{1}{2} + \frac{2}{K} + \frac{1}{2KK_2}, \tag{76}$$

$$C_2 = \alpha_1 + \alpha_2 + \frac{A}{K} - \sqrt{\frac{\left(\alpha_1+\alpha_2+\frac{A}{K}\right)^2}{4} - \left(\alpha_1\alpha_2 + \frac{B}{K}\right)} = \frac{1}{2} + \frac{2}{K} - \frac{1}{2KK_2}; \tag{77}$$

where

$$v_2(t) = K_2(e^{-C_1 t} - e^{-C_2 t}). \tag{78}$$

It is also seen from (53) that

$$\gamma_1 = K\left(\alpha_1 + \alpha_2 + \frac{A}{K}\right) = \frac{K}{2} + 2,$$
$$\gamma_2 = K. \tag{79}$$

Thus, a suitable bound for $\|\mathfrak{W}_2(h)\|$ is obtained. By referring to (56),

$$\begin{aligned}\|\mathfrak{W}_2(h)\| \le M \,|\, &(\gamma_1 - C_1\gamma_2)K_2 e^{-C_1 h} \\ &+ (\gamma_2 C_2 - \gamma_1)K_2 e^{-C_2 h} \,|\, + \max \\ &\cdot \{ \,|\, \gamma_2 K_2(e^{-C_1 h} - e^{-C_2 h} \,|, \\ &|\, (C_1^2\gamma_2 - C_1\gamma_1)K_2 e^{-C_1 h} \\ &+ (\gamma_1 C_2 - \gamma_2 C_2^2)K_2 e^{-C_2 h} \,|\, \}\end{aligned} \tag{80}$$

and (74) is certainly satisfied if

$$M \le 0.15. \tag{81}$$

A substitution of values (67) referred to (75), (76), and (77) yields the following data for K.

K	M
1/10	0.13
1/9.5	0.26
1/8.5	0.39
1/8	0.42

Therefore, it appears that the transient response is definitely bounded in absolute value less than 0.001 for positive K less than or equal to 20.

Conclusion

It has been shown that a generalized weighting function can be developed for a linear pulse-modulated error feedback system which permits a formal solution in closed form as a matrix operator acting on an appropriately defined input vector. Two examples have demonstrated the mathematical procedures involved.

Optimization Based on a Square-Error Criterion with an Arbitrary Weighting Function*

G. J. MURPHY† and N. T. BOLD†

Summary—**Several important criteria for the performance of communication systems and control systems are reviewed, and a new criterion (the mean-weighted-square-error criterion) is then introduced. This is shown to be a special form of a very general criterion proposed earlier, but to have special significance in that it is a generalization of the familiar mean-square-error criterion.**

The minimization of the mean-weighted-square error is treated in detail, and a solution for the optimum physically realizable frequency function of the system is given.

Introduction

In recent years there have been many attempts to establish practical performance criteria for communication systems, but in most cases the measure of system performance has been limited to a specific function of the system error.

Usually, the first step in designing a system for optimum performance is to attribute "importance" to the error as a function of the magnitude of the error, and the next step is to minimize the appropriate function of the error. To provide a background for the discussion of a square-error criterion with an arbitrary weighting function, several well-known criteria are reviewed below, and the concept of "importance of error" is discussed briefly.

The work done in the United States on the statistical anathesis (the word "anathesis" is defined herein to mean "analysis and synthesis") of systems has its origin in the work [1] done by Norbert Wiener in 1942 for the United States government. Wiener's objective was to develop a technique for the analytical design of an optimum linear system, and he established a performance criterion based on the (nondeterministic) input signals received by the system under normal operating conditions. This criterion, referred to as the mean-square-error criterion, is characterized by

$$E_1 \overset{\Delta}{=} \overline{e^2(t)} \tag{1}$$

where

$$e(t) \overset{\Delta}{=} c_d(t) - c_a(t) \tag{2}$$

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is the system error, $c_d(t)$ is the desired response of the system, and $c_a(t)$ is its actual response. That is,

$$E_1 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [c_d(t) - c_a(t)]^2 dt. \tag{3}$$

For deterministic or transient input signals, with

$$e(t) \equiv 0, \qquad t < 0, \tag{4}$$

Hall [2] has proposed the use of

$$E_2 \triangleq \int_0^\infty e^2(t)dt \tag{5}$$

as a measure of system performance.

For application in situations in which (4) is satisfied Graham and Lathrop [3] have proposed the use of the "integral of the time-multiplied absolute error" (ITAE) criterion,

$$E_3 \triangleq \int_0^\infty t\,|\,e(t)\,|\,dt, \tag{6}$$

and Nims [4] has suggested the use of the "integral of the time-multiplied error" (ITE) criterion,

$$E_4(t) \triangleq \int_0^t \tau e(\tau)d\tau. \tag{7}$$

Other integrals [3] that have been used to measure system performance include

$$E_5 \triangleq \int_0^\infty |\,e(t)\,|\,dt, \tag{8}$$

$$E_6 \triangleq \int_0^\infty t^2|\,e(t)\,|\,dt, \tag{9}$$

$$E_7 \triangleq \int_0^\infty te^2(t)dt, \tag{10}$$

$$E_8 \triangleq \int_0^\infty t^2e^2(t)dt \tag{11}$$

and others similar to E_3 but differing from it in the powers of t and $e(t)$ employed.

A more general criterion, suggested by Schultz and Rideout [5], is

$$E_9(\tau) \triangleq \int_0^\infty W(t-\tau)F[e(t,\tau)]dt, \tag{12}$$

where $W(t)$ is an arbitrary weighting function and the error is made to depend on a parameter τ by letting

$$c_d(t) \triangleq r(t-\tau), \tag{13}$$

where $r(t)$ is the input to the system. Schultz and Rideout have investigated the use of the special form

$$E_{10}(\tau) \triangleq \int_0^\infty |\,e(t,\tau)\,|\,dt, \tag{14}$$

and Spooner and Rideout [6] have studied the use of

$$E_{11}(\tau) \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [e(t,\tau)]^2 dt \tag{15}$$

which is also a special form of (12).

The integrand in each of the integrals in (3) and (5)–(11) may be regarded as the importance of the error. For example, the importance of error implied by the use of a square-error criterion is shown in Fig. 1, where it is evident that errors greater than unity are emphasized while errors less than unity are de-emphasized.

As a second example, the importance of error associated with the IAE criterion is shown in Fig. 2. In this case, the importance of the error is directly proportional to the magnitude of the error.

Another important function, suggested by Truxal [7], is shown in Fig. 3. In this case proportional emphasis is placed on error magnitude for errors less than e_0, and all errors of magnitude greater than e_0 are regarded as being equally undesirable.

The Mean-Weighted-Square-Error Criterion

In all the criteria listed above, with the exception of $E_9(\tau)$, some particular weighting of the error is specified. The advantage of $E_9(\tau)$ over the other criteria is, therefore, very great; it is considerably more general and, in fact, it encompasses the other criteria. However, because of its generality, $E_9(\tau)$ is rather difficult to employ to advantage.

A slightly less general criterion, which is readily obtained as a special form of $E_9(\tau)$ by letting $F[e(t,\tau)]$ be $e^2(t)$ and setting τ equal to zero, is

$$E_{12} \triangleq E_9(0) \tag{16}$$

$$= \int_0^\infty W(t)e^2(t)dt, \tag{17}$$

which, from now on, will be referred to as the "integral of the weighted square error" (IWSE) criterion.

The corresponding criterion for optimum anathesis of linear systems with random inputs, instead of deterministic inputs, is

$$E_{13} \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)e^2(t)dt, \tag{18}$$

which will be referred to, from now on, as the "mean-weighted-square-error" criterion. Furthermore, these two criteria encompass all the others listed above except $E_4(t)$ and, of course, $E_9(\tau)$, if $W(t)$ is allowed to be a function of $e(t)$. However, for the purposes of this paper, $W(t)$ is understood to be a function of t alone.

For convenience of notation, let E_{13} be denoted by E. Then the mean-weighted-square-error criterion becomes

$$E = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)e^2(t)dt \tag{19}$$

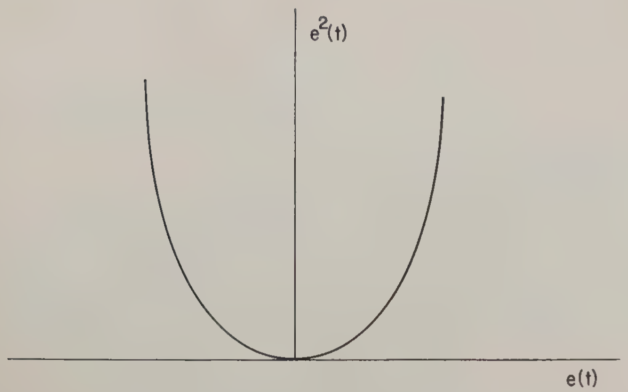

Fig. 1—Error importance function for the error associated with the square-error criterion, $e^2(t)$.

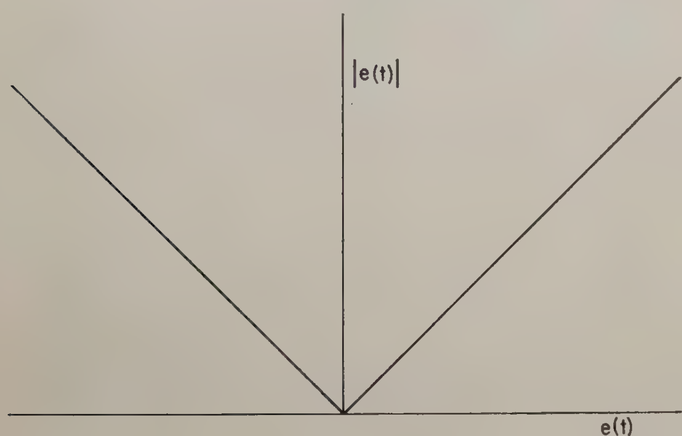

Fig. 2—Error importance function for the error associated with the absolute error criterion $|e(t)|$.

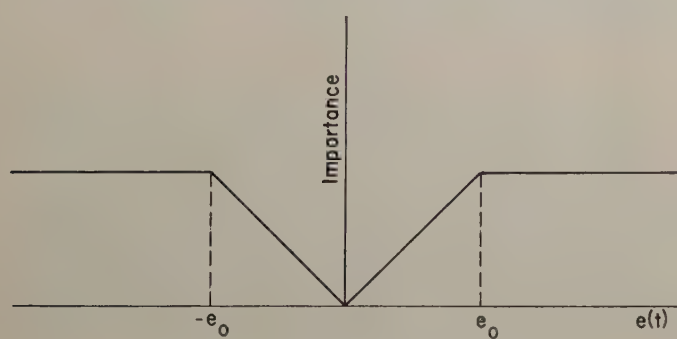

Fig. 3—Saturating error importance function.

and the associated problem is that of determining the optimum system transfer function $G_{\text{opt}}(s)$, for which E has its minimum value for a given $W(t)$.

If the transfer function of the system is denoted by $G(s)$, then, by application of the real convolution theorem,

$$c_a(t) = \int_{-\infty}^{\infty} g(\tau)r(t-\tau)d\tau, \tag{20}$$

where $g(t)$ is the inverse Laplace transform of $G(s)$. Since

$$e(t) \triangleq c_d(t) - c_a(t), \tag{21}$$

substitution of the right-hand member of (20) into (19) yields

$$\begin{aligned} E = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} W(t)\Big[c_d^2(t) &- 2c_d(t)\int_{-\infty}^{\infty} g(\tau)r(t-\tau)d\tau \\ &+ \int_{-\infty}^{\infty} g(\tau)r(t-\tau)\int_{-\infty}^{\infty} g(\gamma)r(t-\gamma)d\gamma d\tau\Big]dt \qquad (22) \\ = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} & W(t)c_d^2(t)dt \\ &- 2\int_{-\infty}^{\infty} g(\tau)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W(t)c_d(t)r(t-\tau)dtd\tau \\ &+ \int_{-\infty}^{\infty} g(\tau)\int_{-\infty}^{\infty} g(\gamma)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \\ &\cdot W(t)r(t-\tau)r(t-\gamma)dtd\gamma d\tau. \qquad (23)[1] \end{aligned}$$

Now let

$$\phi_{wc_dc_d}(\tau,\gamma) \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W(t)c_d(t+\tau)c_d(t+\gamma)dt, \tag{24}$$

$$\phi_{wc_dr}(\tau,\gamma) \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W(t)c_d(t+\tau)r(t+\gamma)dt, \tag{25}$$

and

$$\phi_{wrr}(\tau,\gamma) \triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} W(t)r(t+\tau)r(t+\gamma)dt. \tag{26}$$

If

$$W(t) \equiv 1, \qquad -\infty < t < \infty \tag{27}$$

then

$$\phi_{wc_dc_d}(\tau,\gamma) = \phi_{c_dc_d}(\gamma-\tau), \tag{28}$$

$$\phi_{wc_dr}(\tau,\gamma) = \phi_{c_dr}(\gamma-\tau), \tag{29}$$

and

$$\phi_{wrr}(\tau,\gamma) = \phi_{rr}(\gamma-\tau), \tag{30}$$

where $\phi_{c_dc_d}$, ϕ_{c_dr}, and ϕ_{rr} are the well known auto- and cross-correlation functions of $c_d(t)$ and $r(t)$. Since for arbitrary $W(t)$, $\phi_{wc_dc_d}$, ϕ_{wc_dr}, and ϕ_{wrr} are functions of the two real variables τ and γ, instead of the one real variable $\gamma-\tau$. As for the $W(t)$ defined in (27), the functions defined in (24)–(26) will be referred to as "correlation functions in two-space."

[1] The inversion of the order in which the indicated operations are performed can be justified if the system is stable and $W(t)$, $r(t)$, and the functions defined in (24)–(26) are bounded.

Substitution of the left-hand members of (24)–(26) into (23) yields

$$E = \phi_{wcdcd}(0, 0) - 2\int_{-\infty}^{\infty} g(\tau)\phi_{wcdr}(0, -\tau)d\tau$$
$$+ \int_{-\infty}^{\infty} g(\tau)\int_{-\infty}^{\infty} g(\gamma)\phi_{wrr}(-\tau, -\gamma)d\gamma d\tau \tag{31}$$
$$= \Big[\phi_{wcdcd}(\tau, \gamma) - \int_{-\infty}^{\infty} g(x)\phi_{wcdr}(\tau, \gamma - x)dx$$
$$- \int_{-\infty}^{\infty} g(x)\phi_{wcdr}(\gamma, \tau - x)dx$$
$$+ \int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} g(y)\phi_{wrr}(\tau - x, \gamma - y)dydx\Big]\Bigg|_{\tau=\gamma=0}, \tag{32}$$

where it is evident that the mean-weighted-square error is completely determined when the unit-impulse response of the system and the two-space correlation functions $\phi_{wcdr}(\tau, \gamma)$, $\phi_{wcdcd}(\tau, \gamma)$, and $\phi_{wrr}(\tau, \gamma)$ are specified.

Minimization of the Mean-Weighted-Square Error

The mean-weighted-square error can be minimized by application of variational calculus as follows. Let $\xi(x)$ be a continuous function of x such that

$$\xi(\infty) = 0 \tag{33a}$$

and

$$\xi(x) = 0, \qquad x \leq 0, \tag{33b}$$

and let it be assumed that $g(x)$ is continuous for $x>0$. If $g(x)+\beta\xi(x)$ is substituted for $g(x)$ and $g(y)+\beta\xi(y)$ is substituted for $g(y)$ in the right-hand member of (32), then the left-hand member of (32) becomes $E+\Delta E$. If (32) is subtracted from the equation thus obtained, the result is the variation in E which is thus found to be

$$\Delta E = \Big[-\beta\int_{-\infty}^{\infty} \xi(x)\phi_{wcdr}(\tau, \gamma - x)dx$$
$$- \beta\int_{-\infty}^{\infty} \xi(x)\phi_{wcdr}(\gamma, \tau - x)dx$$
$$+ \beta\int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} \xi(y)\phi_{wrr}(\tau - x, \gamma - y)dydx$$
$$+ \beta\int_{-\infty}^{\infty} \xi(x)\int_{-\infty}^{\infty} g(y)\phi_{wrr}(\tau - x, \gamma - 1y)dydx$$
$$+ \beta^2\int_{-\infty}^{\infty} \xi(x)$$
$$\cdot\int_{-\infty}^{\infty} \xi(y)\phi_{wrr}(\tau - x, \gamma - y)dydx\Big]\Bigg|_{\tau=\gamma=0}. \tag{34}$$

The necessary condition for an extremum of ΔE is

$$\frac{\partial \Delta E}{\partial \beta}\Bigg|_{\beta=0} = 0. \tag{35}$$

From (34) it is found that

$$\frac{\partial \Delta E}{\partial \beta}\Bigg|_{\beta=0}$$
$$= \Big\{-\int_{-\infty}^{\infty} \xi(x)[\phi_{wcdr}(\tau, \gamma - x) + \phi_{wcdr}(\gamma, \tau - x)]dx$$
$$+ \int_{-\infty}^{\infty} g(x)\int_{-\infty}^{\infty} \xi(y)\phi_{wrr}(\tau - x, \gamma - y)dydx$$
$$+ \int_{-\infty}^{\infty} \xi(x)\int_{-\infty}^{\infty} g(y)\phi_{wrr}(\tau - x, \gamma - y)dydx\Big\}\Bigg|_{\tau=\gamma=0}. \tag{36}$$

Changing the order of integration[2] in the first double integral on the right-hand side of (36) yields

$$\frac{\partial \Delta E}{\partial \beta}\Bigg|_{\beta=0}$$
$$= \Big\{-\int_{-\infty}^{\infty} \xi(x)[\phi_{wcdr}(\tau, \gamma - x) + \phi_{wcdr}(\gamma, \tau - x)]dx$$
$$+ \int_{-\infty}^{\infty} \xi(x)\int_{-\infty}^{\infty} g(y)\phi_{wrr}(\gamma - x, \tau - y)dydx$$
$$+ \int_{-\infty}^{\infty} \xi(x)\int_{-\infty}^{\infty} g(y)\phi_{wrr}(\tau - x, \gamma - y)dydx\Big\}\Bigg|_{\tau=\gamma=0} \tag{37}$$
$$= \Big[\int_{-\infty}^{\infty} \xi(x)\Big\{\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y)$$
$$+ \phi_{wrr}(\tau - x, \gamma - y)]dy - \phi_{wcdr}(\tau, \gamma - x)$$
$$- \phi_{wcdr}(\gamma, \tau - x)\Big\}dx\Big]\Bigg|_{\tau=\gamma=0}. \tag{38}$$

According to (35), the right-hand side of (38) must equal zero for arbitrary $\xi(x)$ satisfying (33a) and (33b). It follows that

$$\Big\{\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y) + \phi_{wrr}(\tau - x, \gamma - y)]dy$$
$$- \phi_{wcdr}(\tau, \gamma - x) - \phi_{wcdr}(\gamma, \tau - x)\Big\}\Bigg|_{\tau=\gamma=0} = 0,$$
$$x > 0. \tag{39}$$

[2] A justification of this procedure is given in the Appendix.

Taking the double Fourier transform of both sides of (39) gives

$$\left\{\int_{-\infty}^{\infty} e^{-j\omega'\tau}\int_{-\infty}^{\infty} e^{-j\omega\gamma}\int_{-\infty}^{\infty} g(y)[\phi_{wrr}(\gamma - x, \tau - y) + \phi_{wrr}(\tau - x, \gamma - y)]dyd\gamma d\tau - \int_{-\infty}^{\infty} e^{-j\omega'\tau}\int_{-\infty}^{\infty} e^{-j\omega\gamma}[\phi_{wc_dr}(\tau, \gamma - x) + \phi_{wc_dr}(\gamma, \tau - x)]d\gamma d\tau\right\}\Bigg|_{\tau=\gamma=0} = 0, \quad x > 0 \quad (40)$$

if the integral on the left-hand side of (40) is absolutely convergent. Changing the order of integration[3] then gives

$$\left\{\int_{-\infty}^{\infty} g(y)[\psi_{wrr}(j\omega, j\omega')e^{-j\omega x}e^{-j\omega' y}+\psi_{wrr}(j\omega', j\omega)e^{-j\omega' x}e^{-j\omega y}]dy - e^{-j\omega x}\psi_{wc_dr}(j\omega', j\omega) - e^{-j\omega' x}\psi_{wc_dr}(j\omega, j\omega')\right\}\Bigg|_{\tau=\gamma=0} = 0, \quad x > 0, \quad (41)$$

where ψ_{wrr} is a double Fourier transform of ϕ_{wrr}, and so on.

Eq. (41) can be rewritten in the form

$$\{[G(j\omega')\psi_{wrr}(j\omega, j\omega') - \psi_{wc_dr}(j\omega', j\omega)]e^{-j\omega x} + [G(j\omega)\psi_{wrr}(j\omega', j\omega) - \psi_{wc_dr}(j\omega, j\omega')]e^{-j\omega' x}\}\,|_{\tau=\gamma=0} = 0, \quad x > 0. \quad (42)$$

Taking the double Fourier transform of both sides of (42) gives

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G(j\omega')\psi_{wrr}(j\omega, j\omega') - \psi_{wc_dr}(j\omega', j\omega)]e^{-j\omega x}d\omega d\omega' + \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[G(j\omega)\psi_{wrr}(j\omega', j\omega) - \psi_{wc_dr}(j\omega, j\omega')]e^{-j\omega' x}d\omega d\omega' = 0, \quad x > 0. \quad (43)$$

It can be shown that the first integral on the left-hand side of (43) cannot be the negative of the second integral. Therefore, if a solution exists it is obtained when both integrals are zero. This will be the case if the integrands are identically zero for $x>0$. Thus, it is found that the mean-weighted-square error is minimized if the system frequency function $G(j\omega)$ satisfies

$$G(j\omega')\psi_{wrr}(j\omega, j\omega') = \psi_{wc_dr}(j\omega', j\omega) \quad (44)$$

and

$$G(j\omega)\psi_{wrr}(j\omega', j\omega) = \psi_{wc_dr}(j\omega, j\omega'). \quad (45)$$

[3] The justification for this operation is Fubini's theorem. See Appendix.

The solution to these equations is

$$G_{\text{opt}}(j\omega) = \frac{\psi_{wc_dr}(j\omega, j\omega')}{\psi_{wrr}(j\omega', j\omega)} \quad (46)$$

which serves to define the optimum frequency function for minimization of the mean-weighted-square error.

In a typical situation of considerable interest, the input to the system is the sum of a signal $s(t)$ and noise $n(t)$ which are uncorrelated[4] in the two-dimensional space (τ, γ), and the desired output is

$$c_d(t) = \int_{-\infty}^{t} g_d(\tau)s(t - \tau)d\tau. \quad (47)$$

In this event,

$$\psi_{wc_dr}(j\omega, j\omega') = G_d(j\omega)\psi_{wss}(j\omega, j\omega') \quad (48)$$

and therefore (46) becomes

$$G_{\text{opt}}(j\omega) = \frac{G_d(j\omega)\psi_{wss}(j\omega, j\omega')}{\psi_{wss}(j\omega, j\omega') + \psi_{wnn}(j\omega, j\omega')}. \quad (49)$$

Physical Realizability

The existence of the solution indicated in (46) does not imply physical realizability of the optimum frequency function. Accordingly, the problem of designing the optimum system on the basis of the weighted-mean-square-error criterion is not in general completely solved by application of (46).

The determination of the optimum *physically realizable* frequency function is based on the fact that a transfer function $G(s)$ is physically realizable only if

a) $g(t) \equiv 0, \quad t < 0$

and

b) $g(t)$ is bounded[5] for $t > 0$.

Since the existence of poles of $G(s)$ in the right-hand half of the s plane requires that $g(t)$ differ from zero for $t<0$, it is necessary that no poles of $G(s)$ lie to the right of the imaginary axis in the s plane. Therefore, if the optimum frequency function given by (46) or (47) is a rational function of ω with no poles below the real axis of the ω plane, the optimum frequency function is physically realizable.

If $G_{\text{opt}}(j\omega)$ is a rational function[6] of ω with one or more poles below the real axis in the ω plane, the following ap-

[4] That is,

$$\phi_{wsn}(\tau, \gamma) \equiv \phi_{wns}(\tau, \gamma) \equiv 0.$$

[5] From a practical point of view, this condition can usually be replaced by

$$\lim_{t\to\infty} g(t) = 0.$$

[6] A rational $G(j\omega)$ multiplied by the exponential factor $e^{-\alpha s}$ may be treated similarly.

proach may be used. As in the optimum synthesis procedure based on the rms error criterion, let the ordinary power spectral density of the system input $r(t)$ be

$$\Phi_{rr}(s) = \Phi_{rr}^{+}(s)\Phi_{rr}^{-}(s), \quad (50)$$

where $\Phi_{rr}^{+}(s)$ has poles and zeros only in the left-hand half of the s plane and $\Phi_{rr}^{-}(s)$ has poles and zeros only in the right-hand half of the s plane. The input $r(t)$ is first converted to white noise by passing it through a unit with the physically realizable transfer function

$$G_1(s) = \frac{1}{\Phi_{rr}^{+}(s)} \cdot \quad (51)$$

The white noise must be passed through a physically unrealizable unit with the transfer function

$$G_2(s) = \frac{G_{\text{opt}}(s)}{G_1(s)} \quad (52)$$

to obtain the optimum response. This optimum response is the sum of two statistically independent components [7]: 1) a completely predictable component due to all pulses of the white noise that have previously occurred and the pulse (if one exists) that is simultaneously occurring, and 2) a completely unpredictable component due to pulses of the white noise that will occur in the future.

On the average, the best physically realizable measure of the second of these components is zero. Therefore, the optimum physically realizable frequency function $G_{2or}(j\omega)$ is obtained by making

$$g_{2or}(t) = \begin{cases} 0, & t < 0 \\ \mathfrak{F}^{-1}[\Phi_{rr}^{+}(j\omega)G_{\text{opt}}(j\omega)], & t \geq 0. \end{cases} \quad (53)$$

By combining (53) and (46) it is then found that the optimum physically realizable frequency function $G_{or}(j\omega)$ is

$$G_{or}(j\omega) = \frac{1}{\Phi_{rr}^{+}(j\omega)} \int_0^{\infty} e^{-j\omega t} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{j\omega t} \cdot \frac{\Phi_{rr}^{+}(j\omega)\psi_{wcdr}(j\omega, j\omega')}{\psi_{wrr}(j\omega, j\omega')} d\omega dt. \quad (54)$$

Similarly, for the special case where the system input is the sum of a signal $s(t)$ and noise $n(t)$ which are uncorrelated in the two-dimensional space, and where the desired response is given by (47), it is found that the optimum physically realizable frequency function is

$$G_{or}(j\omega) = \frac{1}{\Phi_{rr}^{+}(j\omega)} \int_0^{\infty} e^{-j\omega t} \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} e^{j\omega t} \cdot \frac{G_d(j\omega)\Phi_{rr}^{+}(j\omega)\psi_{wss}(j\omega, j\omega')}{\psi_{wss}(j\omega, j\omega') + \psi_{wnn}(j\omega, j\omega')} d\omega dt. \quad (55)$$

Limitations

Although (35) is a statement of a necessary condition for minimization of the mean-weighted-square error, it is not sufficient. In order that the extremum resulting from satisfaction of (35) be a minimum, it is necessary in addition that

$$\left.\frac{\partial^2 \Delta E}{\partial \beta^2}\right|_{\beta=0} > 0. \quad (56)$$

In general,

$$\left.\frac{\partial^2 \Delta E}{\partial \beta^2}\right|_{\beta=0} = 2\int_{-\infty}^{\infty} \xi(x) \int_{-\infty}^{\infty} \xi(y)\phi_{wrr}(\tau - x, \gamma - y)dydx \Big|_{\tau=\gamma=0} \quad (57)$$

$$= 2\int_{-\infty}^{\infty} \xi(x) \int_{-\infty}^{\infty} \xi(y) \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \cdot W(t)r(-x)r(-y)dtdydx. \quad (58)$$

Changing the order of integration[7] and averaging yields

$$\left.\frac{\partial^2 \Delta E}{\partial \beta^2}\right|_{\beta=0} = \lim_{T\to\infty} \frac{2}{2T}\int_{-T}^{T} W(t) \int_{-\infty}^{\infty} \xi(x)r(t-x) \int_{-\infty}^{\infty} \cdot \xi(y)r(t-y)dydxdt \quad (59)$$

$$= \lim_{T\to\infty} \frac{2}{2T}\int_{-T}^{T} W(t)m^2(t)dt, \quad (60)$$

where

$$m(t) \triangleq \int_{-\infty}^{\infty} \xi(x)r(t-x)dx. \quad (61)$$

Thus, it is seen that the extremum obtained by satisfying (35) is a minimum if

$$W(t) > 0, \quad -\infty < t < \infty, \quad (62)$$

which is acceptable as a sufficient condition that does not impose a severe limitation on the choice of $W(t)$.

Conclusion

A fairly general new criterion for the performance of communication and control systems has been presented and investigated in detail. A technique for minimizing the mean-weighted-square error (MWSE) has been presented as has been a solution for the optimum physically realizable frequency function. It has been found that the procedure for the use of this criterion is very similar to that for the use of the familiar mean-square-error cri-

[7] The justification for this operation is similar to that used in obtaining (37) from (36).

terion. However, in using the new MWSE criterion, the anathesist has some freedom in the manner in which he may assign importance or undesirability to errors at different instants in time.

Appendix

If $\xi(y)$ is chosen so that

$$\int_{-\infty}^{\infty} g(x) \int_{-\infty}^{\infty} \xi(y)\phi_{wrr}(\tau - x, \gamma - y)dydx \qquad (63)$$

is absolutely convergent, then by Fubini's theorem,[8] the repeated integrals taken in any order exist and are equal. It has previously been stated that $\xi(y)$ is continuous and arbitrary except for the physical realizability restrictions,

$$\xi(y) = 0, \qquad y \leq 0 \qquad (64)$$

and

$$\xi(\infty) = 0. \qquad (65)$$

In addition, it is now understood that only $\xi(y)$ for which (63) is absolutely convergent will be permitted. It follows that then the order of integration may be changed as required.

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[7] J. G. Truxal, "Automatic Feedback Control System Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y., p. 467; 1955.

[8] N. T. Bold, "Optimization Based on an Error Criterion with an Arbitrary Weighting Function," Ph.D. dissertation, Northwestern University, Evanston, Ill.; August, 1958.

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Multiple-Rate Sampled-Data Systems*

LESTER A. GIMPELSON†

Summary—The characterization of multiple-rate sampled-data systems by the ordinary z-transform of single-rate systems is shown. Single-rate sampling, or impulse modulation, of continuous signals is performed by an impulse modulator, M; the sampled, or "starred," function is described by the z-transform. In an analogous manner, a submultiple-rate modulator is introduced; its presence in a branch allows the passage of every nth pulse, or a train of pulses at a submultiple rate; the nomenclature of single-rate systems is continued through the performance of submultiple-rate "starring" of discrete signals and discrete filters. Table I permits starred expressions to be rewritten as functions of the z-transform in closed form. Techniques are shown for the reduction of discrete, and mixed continuous and discrete systems via flow graphs, so that, after the modulators are removed from the feedback loops, the analysis may proceed by standard methods. Representation of single- and submultiple-rate modulation in the s- and z-planes is used to demonstrate that submultiple-rate modulation of discrete signals is analogous to the impulse modulation of continuous signals.

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Introduction

In many applications computing elements form the nucleus of the control, calculation and display operations. Since the computer is communicating with physical components, definite requirements are placed upon it which govern its operational speed. Even after the most judicious programming, more efficient use of the computation capacity of a digital computer may be possible by computing less critical loops less often.

In the wholly discrete control system of Fig. 1(a), each element of the block diagram represents a portion of the computer program. If the contribution of the upper loop is small compared with that of the lower loop, why not compute the upper loop less often? To accomplish this, the output of $H_3(z)$, shown in Fig. 1(b), either is maintained at its previous value for several interations before recomputation, as in Fig. 1(c), or is

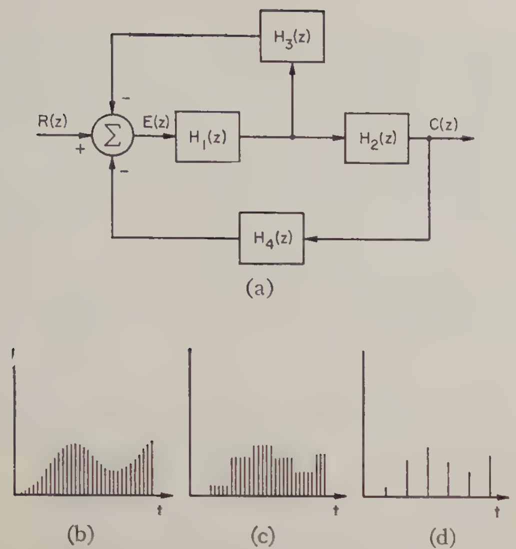

Fig. 1—(a) Block diagram of discrete control system. (b) Output of $H_3(z)$ if computed at every iteration. (c) $H_3(z)$ computed less often; previous output held. (d) $H_3(z)$ computed less often; no output between computations.

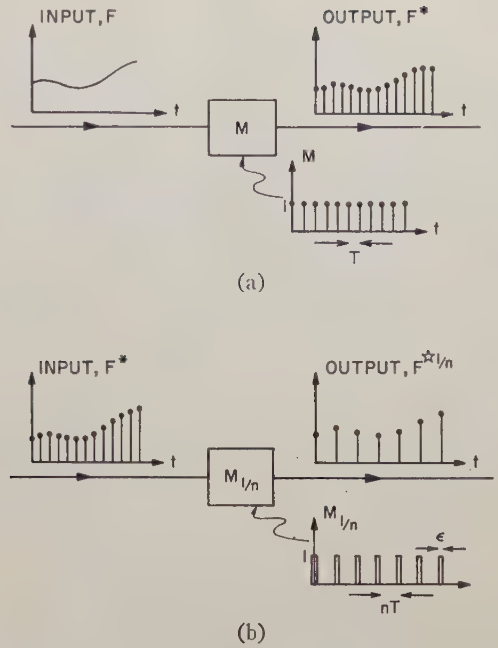

Fig. 2—Sampling devices. (a) Impulse modulator, M, operating at basic system rate. (b) Submultiple-rate modulator, $M_{1/n}$, sampling impulses.

just computed less often without being "held" between computations, as in Fig. 1(d). Now the contribution of the upper loop will be at a lower rate than the basic rate of the system. Assuming that each block requires the same computation time, the determination of the output of $H_3(z)$ at every fifth iteration reduces the average iteration time by almost 20 per cent.

This is but one of a large class of digital systems employing several pulse rates either for economy or because of their basic nature (*e.g.*, radar systems).[1] This paper outlines techniques for the analysis of these systems; in addition, their relation to single-rate systems is shown.

A multiple-rate sampled-data system will be characterized by the presence of synchronous pulses whose rates are harmonically related. The synchronous and harmonic qualifications require that pulses occur at only the basic clock instants and at rates which are submultiples of the basic rate; as will be shown, there are frequently approximations which permit these restrictions to be relaxed.

Sampling Operations

Sampling is accomplished by an impulse modulator, M [Fig. 2(a)], which transmits pulses (approximated by impulses) whose heights are equal to the instantaneous magnitude of the function being sampled. The output of the modulator, $f^*(t)$ ["$f(t)$ starred"], is obtained through the multiplication of the input signal by a train of unity area impulses separated by T seconds. When considered in the frequency domain, the impulse starring of $F(\omega)$ produces sidebands[2] at intervals of ω_s, the sampling rate (Fig. 3). In the s-plane $F^*(s)$ is characterized by an infinite string of poles for each pole of $F(s)$. For example, when the exponential function shown in Fig. 4(a),

$$F(s) = \frac{1}{s+a},$$

is sampled, the single pole at $s=-a$, is replaced by an infinite number of poles [Fig. 4(b)] located at $s=nj\omega-a$:

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{n=\infty}\frac{1}{s-nj\omega+a}.$$

These observations follow from the two equivalent frequency domain expressions for $f^*(t)$,

$$F^*(s) = \sum_{n=-\infty}^{n=\infty} f(nT)\epsilon^{-snT},$$

or,

$$F^*(s) = \frac{1}{T}\sum_{n=-\infty}^{n=\infty} F(s+jn\omega),$$

where T is the sampling period.[3,4] The difficulty, which

[1] G. M. Kranc, "Input-output analysis of multirate feedback systems," IRE Trans. on Automatic Control, vol. AC-3, pp. 21–28; November, 1957. Also, "Multi-Rate Sampled Systems," Columbia University, New York, N. Y., Rept. No. T-14/B; May, 1958.
G. M. Kranc, "Additional techniques for sampled-data feedback problems," 1957 WESCON Convention Record, pt. 4, pp. 157–165.

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[3] B. Widrow, class notes for "Pulsed-Data Systems," Course No. 6.54, Mass. Inst. Tech., Cambridge.
J. G. Truxal, "Automatic Feedback Control System Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 500–558; 1955.

[4] If $f(t)$ is nonzero at $t=0^+$, the right side of the second equation for $F^*(s)$ should be appended by the term $\frac{1}{2}f(0^+)$; in most systems, however, this addition is unnecessary.

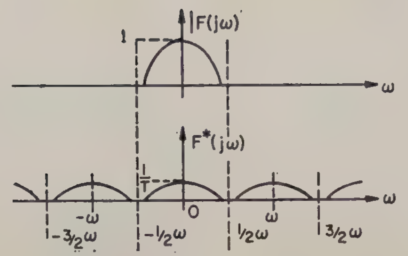


Fig. 3—Production of sidebands by impulse modulation.

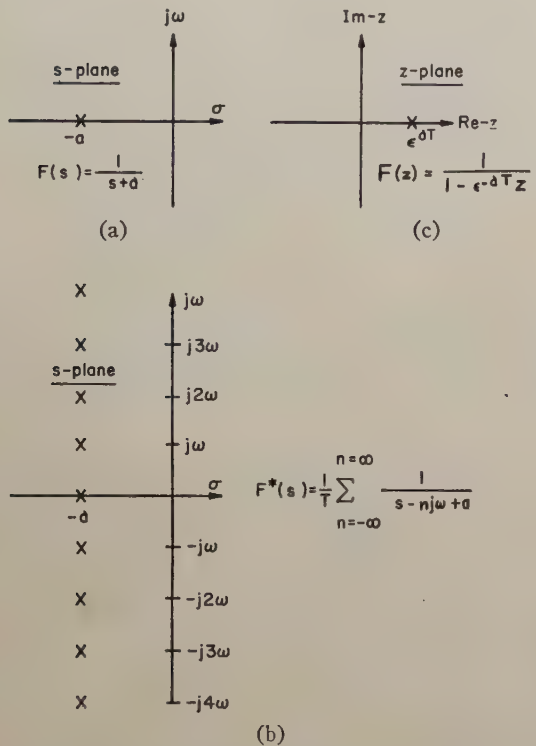


Fig. 4—Frequency domain view of sampling. (a) Single pole of $F(jw)$. (b) Pole plot of $F^*(jw)$ showing multiplicity of poles. (c) $F^*(z)$ in z-plane.

results from infinite numbers of poles, is overcome by the substitution,

$$z = \epsilon^{-sT}$$

which allows $F^*(s)$ to be written as

$$F(z) = \sum_{n=-\infty}^{n=\infty} f(nT)z^n.$$

For positive time only,

$$F(z) = f(0)z^0 + f(T)z^1 + f(2T)z^2 + f(3T)z^3 + \cdots,$$

which is a description of the sampled function at only the sampled instants. (The z-transform, as defined above, differs from the notation used by some authors. The use of a negative exponent permits z to be considered as a unit of delay; expansions are then written in powers of z, which is very convenient in the following material. Expressions using the alternate definition, $z = \epsilon^{+sT}$, are the same as those obtained here with every z replaced by z^{-1}.)

The z-transform maps the left half of the s-plane onto the area outside[5] of the unit circle in the z-plane; the unit circle corresponds to the $j\omega$-axis; in general, circles concentric about the origin of the z-plane correspond to lines of constant σ, where σ is the parameter of the horizontal axis of the s-plane. Thus, system characteristics may be analyzed in the z-plane with techniques similar to those employed in the s-plane.

Representation of a signal occurring at a rate which is $(1/n)$th of the basic rate is possible using powers of z. For example,

$$\frac{z}{1 - bz} = z[(bz)^0 + (bz)^1 + (bz)^2 + \cdots],$$

becomes

$$\frac{b^{n-1}z^n}{1 - (bz)^n} = b^{n-1}z^n[(bz)^0 + (bz)^n + (bz)^{2n} + \cdots],$$

when the original function is resampled at every nth instant, or at a submultiple rate.

Resampling is performed by a submultiple-rate modulator [Fig. 2(b)], which multiplies the input signal by a train of finite width, unity height pulses, separated by nT seconds.[6,7] The output of the submultiple-rate modulator is termed $F^{\star 1/n}$ to indicate the reduced sampling rate. The expressions above can be written as

$$\left[\frac{z}{1 - bz}\right]^{\star 1/n} = \frac{b^{n-1}z^n}{1 - (bz)^n}.$$

Thus, there are two types of starring operations: impulse and submultiple-rate. Tables of z-transforms[8] are useful in the evaluation of impulsively starred functions; determination of submultiply starred functions[5] is simplified by the use of Table I for which starred functions are expanded into partial fractions of the form

$$\left[\frac{z^k}{(1 - bz)^d}\right]^{\star 1/n},$$

where k is the number of units of delay; d governs the multiplicity of the poles; $1/b$ is the location of the poles (possibly complex); and $1/n$ indicates the submultiple-rate sampling. (The development of this table, instruc-

[5] The convention employed here defines $z = \epsilon^{-sT}$.

[6] The use of a multiple-rate modulator and the "family" of modulation processes are based upon suggestions by Prof. B. Widrow, Mass. Inst. Tech., Cambridge.

[7] More rapid samplers are inserted in single-rate networks by several authors in order to obtain information concerning the operation of these systems between the sampling instants:
W. K. Linvill and R. M. Sittler, "Extension of conventional techniques to the design of sampled-data systems," 1953 WESCON CONVENTION RECORD, pt. 1; 1953.
G. V. Lago and J. G. Truxal, "The design of sampled-data feedback systems," *Trans. AIEE*, vol. 70, pt. 2, pp. 247–253; November, 1954.

[8] R. H. Baker, "The pulse transfer function and its application to sampling servo systems," *Proc. IEE (London)*, vol. 99, pt. 4, pp. 302–317; December, 1952.
E. I. Jury, "Analysis and synthesis of sampled-data control systems," *Trans. AIEE*, vol. 73, pt. 1, pp. 332–346; September, 1954.
E. I. Jury, "Additions to the modified z-transform method," 1957 WESCON CONVENTION RECORD, pt. 4, pp. 136–156.

TABLE I
SUBMULTIPLE-RATE EQUIVALENTS[5]

$d=1$	$\dfrac{b^{dn-k}z^{dn}}{[1-(bz)^n]^d}$
$d=2$	$\dfrac{[n-(k-1)]b^{(d-1)n-k}z^{(d-1)n}+[k-(d-1)]b^{dn-k}z^{dn}}{[1-(bz)^n]^d}$
$d=3$	$\dfrac{\sum\limits_{n-(k-1)} b^{(d-2)n-k}z^{(d-2)n}+\Big(\sum\limits_{k+n-(d-1)}-d\sum\limits_{k-(d-1)}\Big)b^{(d-1)n-k}z^{(d-1)n}+\sum\limits_{k-(d-1)} b^{dn-k}z^{dn}}{[1-(bz)^n]^d}$
$d=4$	$\dfrac{\sum\sum\limits_{n-(k-1)} b^{(d-3)n-k}z^{(d-3)n}+\Big(\sum\sum\limits_{2n-(k-1)}-d\sum\sum\limits_{n-(k-1)}\Big)b^{(d-2)n-k}z^{(d-2)n}+\Big(\sum\sum\limits_{k+n-(d-1)}-d\sum\sum\limits_{k-(d-1)}\Big)b^{(d-1)n-k}z^{(d-1)n}+\sum\sum\limits_{k-(d-1)} b^{dn-k}z^{dn}}{[1-(bz)^n]^d}$

tions for its use and examples of equivalents with real and with complex poles are included in the Appendix.)

NATURE OF MULTIPLE-RATE SYSTEMS

The use of this new modulator and Table I will be illustrated through a simple example. The system in Fig. 5 consists of two delay loops; in the flow graph solid lines indicate pulses at the basic rate, while the dotted branch experiences pulses at the submultiple rate only, governed by the closure rate of the switch. The transmission of the system is $1/(1-z)$, an integrator, for closure of the switch at each basic clock instant. If the switch were to close at every third basic clock instant, the pulse rate at node a would be one-third of the basic rate. By considering the transmission of signal through the network, the response of the system to a pulse coincident with a switch closure, and a closure rate of one-third, is readily obtained as[5]

$$T_0 = 1+\tfrac{1}{2}z+\tfrac{1}{4}z^2+\tfrac{1}{4}z^3+\tfrac{1}{8}z^4+\tfrac{1}{16}z^5+\tfrac{1}{16}z^6+\tfrac{1}{32}z^7+\cdots.$$

Arrival of the input pulse at one iteration after a switch closure leads to

$$T_1 = 1+\tfrac{1}{2}z+\tfrac{1}{2}z^2+\tfrac{1}{4}z^3+\tfrac{1}{8}z^4+\tfrac{1}{8}z^5+\tfrac{1}{16}z^6+\tfrac{1}{32}z^7+\cdots.$$

Also,

$$T_2 = 1+z+\tfrac{1}{2}z^2+\tfrac{1}{4}z^3+\tfrac{1}{4}z^4+\tfrac{1}{8}z^5+\tfrac{1}{16}z^6+\tfrac{1}{16}z^7+\cdots.$$

In closed form, the impulse responses are,

$$T_0 = \frac{1+\frac{1}{2}z+\frac{1}{4}z^2}{1-\frac{1}{4}z^3}$$

$$T_1 = \frac{1+\frac{1}{2}z+\frac{1}{2}z^2}{1-\frac{1}{4}z^3}$$

$$T_2 = \frac{1+z+\frac{1}{2}z^3}{1-\frac{1}{4}z^3}.$$

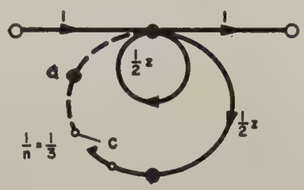


Fig. 5—Multiple-rate system.

These responses show three important characteristics of multiple-rate sampled systems: they have time-varying impulse responses, which are a consequence of the system's cyclic changes in structure; the system's degree is increased (above responses are third-order, while the single-rate system had a first-order response); the poles of each response are always the same, but the numerators are not unique, indicating that while the stability and general character of the system is not a function of the timing of an input, the exact nature of the response is time-varying.

As Table I shows,

$$\left[\frac{z^k}{(1-bz)^d}\right]^{*1/n} = \frac{(\cdots)}{[1-(bz)^n]^d},$$

the denominators of all submultiply starred functions undergo the same transformations, only dependent upon the values of n and d. Prior to starring, poles of fractions of the form $z/(1+bz)^d$ are located at $1/b$, and their multiplicity is equal to the power, d, to which the denominator is raised. After starring, the multiplicity is still governed exclusively by d, however; the locations become a function of n. There are now n poles, equally spaced about a circle of radius $1/b$; see Fig. 6. (Since points which are equidistant from the origin of the z-plane, lie on lines of constant σ in the s-plane, submultiple-rate sampling cannot affect a system's settling characteristics; rise time and overshoot are, however, a function of the lower sampling rate.) If d were increased to 2, each pole would be double order; if $d=3$, triple order, etc. Clearly this is so since the modulator cannot

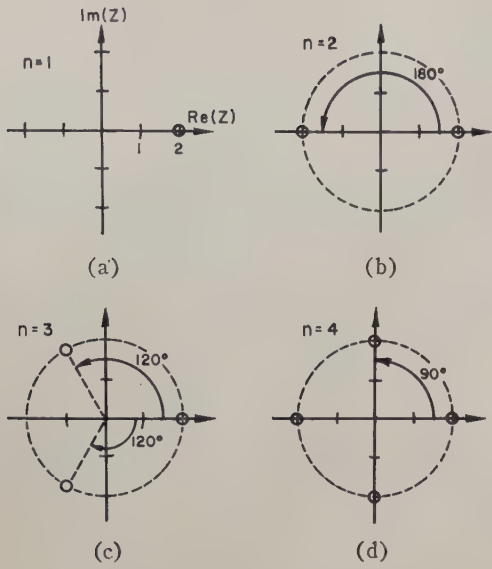


Fig. 6—Zeros of $[1-(\frac{1}{2}z)^n]$.

change the fundamental form of the signal it passes, and therefore the multiplicity cannot be governed by n.

Impulse modulation causes each pole of the continuous function to be repeated an infinite number of times in the s-plane [Fig. 4(b)]. The infinite number of initial conditions which would seem to be required is explained by the infinite number of ways of phasing the impulse modulator with respect to the continuous signal. The application of the z-transform coalesces this multitude of singularities into a single pole in the z-plane for each infinite string of poles in the s-plane [Fig. 4(c)]. When a sampled function is again sampled, as by a submultiple-rate modulator, the resulting function will have a higher-order representation in the z-plane (Fig. 6) and a multiply infinite number of poles in the s-plane. The effect of submultiple-rate sampling is therefore very similar to the single-rate sampling of a continuous signal, since the occurrence of new poles in the z-plane has the same significance as the development of infinite numbers of poles in the s-plane. Due to the qualification here that the sampling rates be submultiples of the basic rate, the number of possible phasings of the modulators with respect to external discrete signals is limited, and therefore the number of additional poles is finite in the z-plane. These poles distribute evenly about circles whose radii are those of the original poles. (If sampling rates are not commensurate, there will be a continuous distribution of poles about these circles.)

To illustrate these points, consider an integrator which has the transforms[5] $1/s$ and $1/(1-z)$. If submultiply sampled at $(1/n)$th of the basic rate, the transform becomes $1/(1-z^n)$. This is still a first-order function; however, the sampling is less often. A system incorporating this element together with its modulator has n impulse responses, or alternately, there are n possible phasings of the submultiple-rate modulator. If there are storage elements in a single-rate system, the degree of the system will be equal to their number (*e.g.*, $1/s$ and $1/(1-z)$, or $1/s^2$ and $zT/(1-z)^2$). The addition of a submultiple-rate, $1/n$, increases the function's order to n times the original order (*e.g.*, $1/s$ and $1/(1-z^n)$, or $1/s^2$ and $z^nT/(1-z^n)^2$). Inclusion of several modulators results in increases proportional to the product of the n_i designators.

The broad areas of similarity in the mathematics and the interpretation of these operations leads to the development of a hierarchy of modulations: namely, impulse modulation is to continuous signals as submultiple-rate modulation is to discrete signals.

Reduction Techniques

Returning to the system of Fig. 5, the switch will be replaced by a submultiple-rate modulator, passing pulses at one-third of the basic rate. The flow graph of this completely digital system [Fig. 7(a)] is reduced by the selection of node a as the "discrete node" since pulses occur there at the lower rate only. The flow-graph of Fig. 7(b) is obtained by noting 1) the transmission with the branch containing $M_{1/3}$ removed, $1/(1-\frac{1}{2}z)$; 2) the transmission from the input to node a, $[\frac{1}{2}z/(1-\frac{1}{2}z)]M_{1/3}$; 3) the transmission out of node a, $1/(1-\frac{1}{2}z)$; and 4) the self-loop about that node, $[\frac{1}{2}z/(1-\frac{1}{2}z)]M_{1/3}$.

The presence of modulators before and after a continuous element permits impulse starring of the element and the elimination of the second modulator. In the same manner, a discrete filter placed between two submultiple-rate modulators is submultiply starred together with the removal of either modulator. The function in the self-loop of Fig. 7(b) meets this criterion, and starring is therefore permissible, thus removing the modulator from the feedback loop. The transmission is now

$$T = \frac{1}{1-\frac{1}{2}z}$$
$$+\left[\frac{\frac{1}{2}z}{1-\frac{1}{2}z}\right]M_{1/3}\cdot\frac{1}{1-\left[\frac{\frac{1}{2}z}{1-\frac{1}{2}z}\right]^{*1/3}}\cdot\frac{1}{1-\frac{1}{2}z}.$$

Evaluation of the starred function leads to

$$T = \frac{1}{1-\frac{1}{2}z}+\left[\frac{\frac{1}{2}z}{1-\frac{1}{2}z}\right]M_{1/3}\left(\frac{1-\frac{1}{8}z^3}{1-\frac{1}{4}z^3}\cdot\frac{1}{1-\frac{1}{2}z}\right).$$

Note that the ordering of the terms in this expression carries the same significance as in single-rate sampled-data systems. The first term indicates the signal which feeds through the network irrespective of the modulators; the input signal is filtered by the term in square brackets and its output is sampled at every third iteration; finally, the one-third rate signal is filtered by the term in parentheses. (At this point it is evident that the system is absolutely stable, as every element in the forward branches is stable, and the modulators cannot, of themselves, introduce an instability into a forward path.)

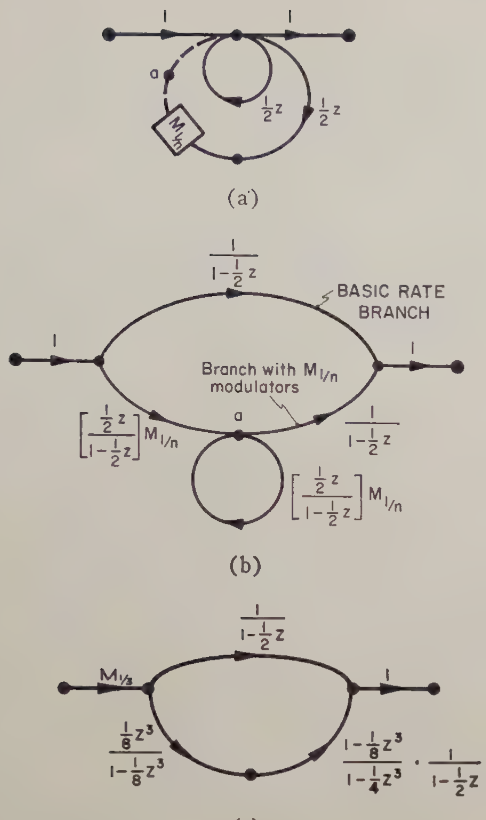


Fig. 7—Reduction of Fig. 4.

Since any representative[9] impulse response will provide the system's pole configuration (proof of this point will follow), it is only necessary to determine one convenient response, usually T_0. When the input is a single pulse at $t=0$, the system is unaffected by the placement of a modulator in the input branch [Fig. 7(c)]. This addition allows the branch joining the input with node a to be starred, and the response is obtained as

$$T_0 = \frac{1}{1-\frac{1}{2}z} + \frac{\frac{1}{8}z^3}{1-\frac{1}{8}z^3}\cdot\frac{1-\frac{1}{8}z^3}{1-\frac{1}{4}z^3}\cdot\frac{1}{1-\frac{1}{2}z}$$
$$= \frac{1+\frac{1}{2}z+\frac{1}{4}z^2}{1-\frac{1}{4}z^3}.$$

If the input, $R(z)$, were at the lower rate, the output is simply

$$C(z) = R(z)\cdot T_0(z),$$

at the lower rate. Through a change of viewpoint, networks which include a higher sampling rate are analyzed by considering the fast rate as a pseudo-basic rate, and the actual basic rate as a pseudo-submultiple rate.

Reduced flow graphs, as in Fig. 7(b), are complete descriptions of the original system's terminal characteristics at the sampling instants. The cyclic, time-varying nature of the impulse responses allows investigation of any T_j, since an input at time t_i can be considered as contributing to initial conditions for the response T_j at some later time, t_j. Therefore, the poles of these responses cannot depend upon the timing of the input. Alternately, by removing the modulators from the feedback loops, this method of reduction will yield flowgraphs composed only of forward path transmissions, T_s's. The total transmission, T_j, is the sum of the T_s's. In this summation, the poles of the T_s's contribute to the poles of T_j; none of the numerators of the T_s's, which are the only functions dependent upon the input, can affect the denominator of T_j. These two demonstrations of the nontime-varying character of the poles of the impulse responses permit use of any T_i (usually T_0, as in the example above) for analysis. In addition, this entire analysis technique employs only the ordinary z-transform; as a result, all of the methods of analysis (root locus, Nyquist diagrams, etc.) previously developed for sampled systems, are readily applicable.

As a further example, the wholly digital system of Fig. 1 will be reconsidered; the flow graph, Fig. 8(a), shows two integrators in tandem, the first one providing the necessary delay for both feedback loops.[10] If, for the moment, the modulator is shorted, the impulse response is

$$T_a = \frac{abz}{z^2(1-ad)+z(abk+ad-2)+1}.$$

Elimination of the upper loop replaces T_a with

$$T_b = \frac{abz}{z^2+z(abk-2)+1},$$

which has both of its poles on the unit circle in the z-plane, indicating that any input would lead to an indefinitely persistent output. This system is therefore one which is naturally oscillatory, receiving corrections when the switch in the upper loop closes.

Reduction about node a leads to Fig. 8(b), where the modulator has been removed from the self-loop by submultiple-rate starring. T_0 is then obtained as

$$T_0 = \left[\frac{abz}{z^2+z(abk-2)+1}\right]\left[\frac{z^4+z^2K_1+1}{z^4(1-ad)+z^2K_2+1}\right],$$

if the upper loop sampling rate is one-half of the basic sampling rate; K_1 and K_2 are constants. In all cases where $1/n\neq 1$, the presence of two poles on the unit circle is shown in T_0. The expression in the first parenthesis is exactly the same as T_b, obtained earlier for the transmission with the upper loop deleted. The second parenthesis, which is similar in form to the response T_a,

[9] Degenerate situations sometimes arise. If, for example, $M_{1/n}$ is located at the input, T_0 will be the only finite transmission. The other transmissions have the same denominators, but are considered to have zero numerators.

[10] Each loop in a sampled system requires a delay to prevent the return of a signal to an element during the same iteration. If each loop were not so designed, the approximation of pulses by impulses would not be valid.

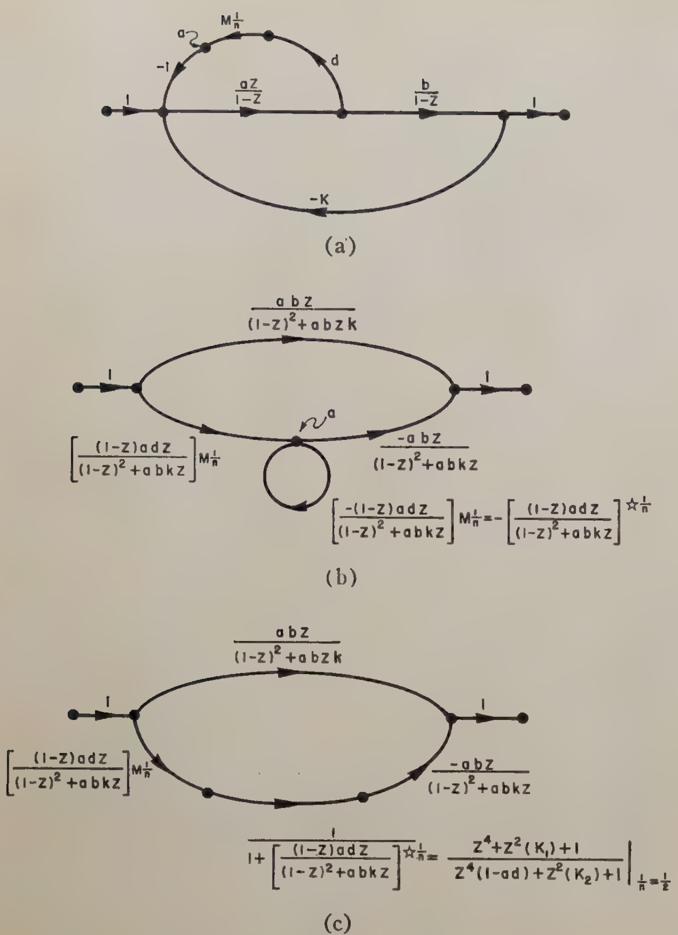


Fig. 8—Reduction of Fig. 1.

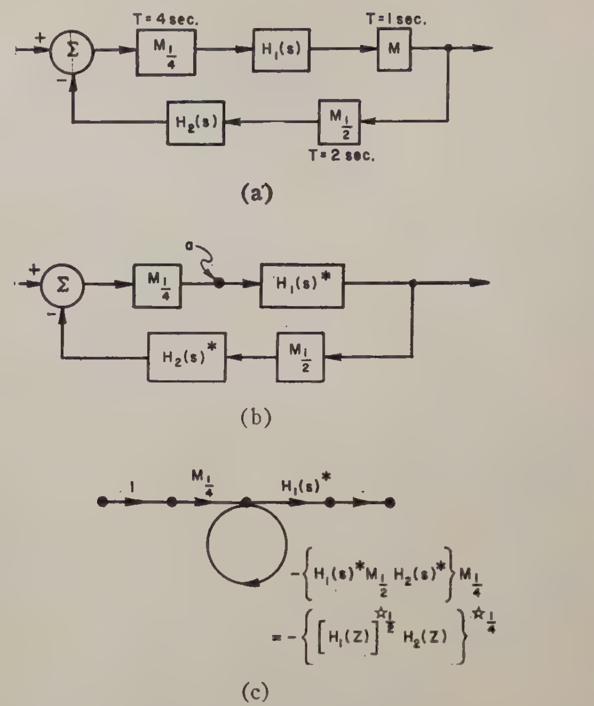


Fig. 9—Mixed system with three rates.

shows the effect of the correction of the upper loop at every nth sampling instant. Intuitive interpretations of this type are frequently possible.

Continuous and Discrete Elements; Approximations

When continuous components occur together with discrete elements operating at different rates, the most rapid sampler determines the basic pulse rate; all others are designated as submultiple-rate pulses. After the continuous elements have been individually starred at the basic rate, the resulting network is wholly discrete (except possibly at the terminal branches), and the analysis is similar to the previous two examples. This approach is illustrated in Fig. 9, which shows a control system whose output pulse rate is four times as rapid as the input rate; in addition, the feedback element is pulsed half as often as the output. Placing impulse modulators at the inputs and outputs of the continuous elements, $H_1(s)$ and $H_2(s)$, is redundant since these modulators operate more frequently than the $M_{1/n}$'s; obviously a continuous element between two impulse or submultiple-rate modulators can be impulsively starred, as in Fig. 9(b), in which the redundancies have been removed. Node a is selected since pulses at this point are at the lowest rate in the circuit, and reduction to a single branch is obtained. The term

$$\{H_1(s)^* \cdot M_{1/2} \cdot H_2(s)^*\} M_{1/4},$$

is evaluated by impulse starring of continuous functions,

$$\{H_1(z) \cdot M_{1/2} \cdot H_2(z)\} M_{1/4},$$

and then by submultiple-rate starring.

$$\{[H_1(z)]^{\star 1/2} \cdot H_2(z)\}^{\star 1/4}.$$

Since the element $H_1(z)$ in the self-loop is preceded by $M_{1/4}$ and followed by $M_{1/2}$, insertion of another $M_{1/2}$ between $M_{1/4}$ and $H_1(z)$ is redundant, and starring at $1/n = \frac{1}{2}$ is valid.[11]

The inclusion of modulators whose rates are not integral multiples of one another is representable by these techniques, but reductions with the complete elemination of modulators is not always possible. In these cases approximations can frequently be made. For example, when a continuous filter is preceded by $M_{1/3}$ and followed by $M_{1/8}$, the latter modulator might be replaced by $M_{1/9}$; of course, these substitutions are only valid when there is a continuous filter separating the modulators, and, even then, care should be exercised in these approximations.

[11] In general, when a filter is followed by a modulator whose rate is a multiple of the input signal rate, the filter may be starred at the more rapid rate.

Conclusion

Systems which lend themselves to representation by a multiple-rate sampled-data system model can be analyzed with facility by the methods developed above. These systems are characterized by the ordinary z-transform, using an extended "starring" technique. A table for the evaluation of submultiply starred functions (with either real or complex poles), as closed form expressions in z, allows the removal of modulators from the feedback loops. The result is a network composed of forward transmissions which may be analyzed by conventional methods.

The stability and settling characteristics of an element are not affected by starring since the new poles produced by this operation always remain on a circle of the same radius as that of the original unstarred function. It is seen that the evaluation of one representative response of the time-varying system will provide sufficient information for most analyses. The demonstration that impulse modulation of continuous signals, resulting in the multiplication of poles in the s-plane, is analogous to the multiple-rate modulation of discrete signals resulting in a multiplication of poles in the z-plane, leads to a unified description of these modulation or sampling operations.

Appendix

The development of Table I[5] was motivated by the need for a ready means of obtaining reduced rate and closed form equivalents of functions of z. For example, when

$$\frac{1}{1-\frac{1}{2}z} = 1 + \tfrac{1}{2}z + (\tfrac{1}{2}z)^2 + (\tfrac{1}{2}z)^3 + \cdots$$

is sampled at one-third of the basic rate, pulses are obtained at the 0th, 3rd, 6th, 9th, $\cdots$, sampling instants; their heights are 1, $(\frac{1}{2})^3$, $(\frac{1}{2})^6$, $(\frac{1}{2})^9$, $\cdots$, respectively. Therefore,

$$\left(\frac{1}{1-\frac{1}{2}z}\right)^{\star 1/3} = 1 + (\tfrac{1}{2}z)^3 + (\tfrac{1}{2}z)^6 + \cdots = \frac{1}{1-(\frac{1}{2}z)^3}.$$

Attempts to obtain a single expression for equivalents, using a generic d, led to prohibitively lengthy terms. However, rather economical formulas were obtained for specific values of d up to $d=4$, which should be sufficient for almost all applications. The formulas were arrived at by long division of the general fraction $z^k/(1-bz)^d$, selection of terms, and arrangement into closed form. (The generic d has been left in Table I to show its form for future extensions.)

To use Table I, the function is expanded into partial fractions and substitutions are made rigorously. The summation notation should be interpreted as follows:

$$\begin{array}{cccccl}
\# & 1 & 2 & 3 & 4 & \\
\hline
 & & 1 & 2 & 3 & \rightarrow \# \\
 & & & 1 & 2 & \\
 & & & & 1 & \\
\hline
 & 1 & 3 & 6 & 10 & \rightarrow \sum_{\#} \\
 & & 1 & 3 & 6 & \\
 & & & 1 & 3 & \\
 & & & & 1 & \\
\hline
 & 1 & 4 & 10 & 20 & \rightarrow \sum\sum_{\#}.
\end{array}$$

For example,

$$\sum\sum_4 = 20, \qquad \sum\sum_2 = 4.$$

Evaluation of a function proceeds as follows:

$$\left[\frac{z^4}{(1-bz)^3}\right]^{\star 1/4} = \frac{\sum_1 b^0 z^4 + \left(\sum_6 - 3\sum_2\right) b^4 z^8 + \sum_2 b^8 z^{12}}{[1-(bz)^4]^3} = \frac{z^4 + 12b^4 z^8 + 3b^8 z^{12}}{[1-(bz)^4]^3}.$$

Note the simple check that the sum of the coefficients of the numerators equals $n^{(d-1)}$; in the example above, $\sum(\text{coef.}) = 16 = 4^{(3-1)}$.

Another example illustrates the determination of a submultiple-rate equivalent of a function with complex roots:

$$T = \frac{(1-\frac{1}{2}z)2z}{\frac{1}{2}z^2 - z + 1} = \frac{z}{1-(\frac{1}{2}+\frac{1}{2}j)z} + \frac{z}{1-(\frac{1}{2}-\frac{1}{2}j)z}.$$

The b's are $(\frac{1}{2}+\frac{1}{2}j)$ and $(\frac{1}{2}-\frac{1}{2}j)$:

$$T^{\star 1/2} = \frac{(\frac{1}{2}+j\frac{1}{2})z^2}{1-(\frac{1}{2}+j\frac{1}{2})^2 z^2} + \frac{(\frac{1}{2}-j\frac{1}{2})z^2}{1-(\frac{1}{2}-j\frac{1}{2})^2 z^2} = \frac{(1-\frac{1}{4}z^2)z^2}{\frac{1}{4}z^4+1}$$

Acknowledgment

The author is most grateful to Prof. B. Widrow, who supervised this investigation, for his advice, assistance, encouragement and many suggestions.

Automatic Control of Three-Dimensional Vector Quantities—Part 2*

A. S. LANGE†

Summary—**In Part 1 of this paper[13] a vector algebra was developed using a three-element column matrix to represent the vector, and a three-by-three matrix to represent a vector transformation operator. Problems in spherical trigonometry were analyzed with the use of a position vector, and the design of automatic computers to solve such problems was considered. In Part 2, the angular velocity vector is introduced for the purpose of analyzing and designing geometric stabilization systems.**

V. Geometric Stabilization

GEOMETRIC stabilization is the process of isolating some controlled member, such as a gun or a radar antenna, from the motion of the platform on which it is mounted, such as a ship, an airplane, or a tank. The geometric stabilization system consists of a device to measure the motion of the base, and servomechanisms which move the controlled member with respect to the platform. The stabilization system functions in such a way that the motion produced by the servos is opposite in sense to that of the platform, so that the net motion of the controlled member with respect to some given set of reference coordinates is zero. In general, the platform has three degrees of (angular) freedom, so that it is necessary to have two or more degrees of freedom in mounting the controlled member on the platform. These several degrees of freedom are achieved by a series of gimbals, so that we speak of a three-gimbal mount or a three-axis mount, since each gimbal is designed so as to rotate with respect to the member on which it is mounted. Several of the more commonly used two- and three-axis antenna mounts are illustrated in Cady, *et al.*[14] The six ways that the three axes of a three-gimbal mount can be arranged are shown in Fig. 6. Associated with each gimbal is a servomechanism, so that each gimbal can apply one restraint to the controlled member. When the controlled member is a gun or a telescope, there is a line associated with it (*viz.*, the barrel of the gun, or the optical axis of the telescope) called the controlled line. In many instances, it is only necessary to isolate the controlled line from the motion of the base, rather than the controlled member itself. For this purpose, a two-axis mount is sufficient, since it is usually not required that a gun barrel not rotate about its own axis. However, if the controlled member itself must be irrotational with respect to the reference frame, then three degrees of freedom are required.

The requirement that the controlled member be isolated from the motion of the base on which it is mounted may be expressed by the vector equation

$$\overline{W}_{ia} = 0 \tag{29}$$

where $\overline{W}_{ia}$ = angular rate of S_a with respect to S_i; S_a is a set of coordinates fixed to the controlled member, and S_i is a set of reference coordinates. It may be recalled that in the previous section, it was necessary to use position vectors in order to describe angular quantities, because angles do not obey the rules of the usual vector algebra. For our purposes, however, angular rates follow these rules, and hence may be treated as vector quantities, whose direction is given by the right-hand rule, and whose length is given by the magnitude of the rotation.[15]

Eq. (29), then, states that the angular velocity of S_a (a coordinate set which defines the controlled member) with respect to S_i (a coordinate set which defines the reference frame) must be zero. Eq. (29) may be written as

$$\overline{W}_{ia} = \overline{W}_{id} + \overline{W}_{da} = 0$$

or

$$\overline{W}_{da} = -\overline{W}_{id} \tag{30}$$

where $\overline{W}_{id}$ is the angular velocity of S_d (a coordinate set which defines the platform on which the controlled member is mounted) with respect to S_i, and $\overline{W}_{da}$ is the angular velocity of S_a with respect to S_d. Eq. (30) shows that if the controlled member, defined by S_a, is to be stationary with respect to the reference frame S_i, then the angular velocity of the controlled member with respect to the platform $\overline{W}_{da}$, must be equal to the angular velocity of the platform S_d with respect to S_i, $\overline{W}_{id}$.

One of the design problems associated with base motion isolation systems is the arrangement or order of the gimbals. In some cases, the designer has no choice in selecting the configuration; however, some aspects of the design which may influence the choice of gimbal sequence will be discussed in Section VIII.

The designer of an automatic control system is also concerned with defining such servo characteristics as bandwidth, maximum velocity, maximum acceleration, gear ratio, torque, and horsepower. In the following discussion, the determination of the angular velocity re-

* Manuscript received by the PGAC, August 10, 1959.
† Systems Div., Bendix Aviation Corp., Ann Arbor, Mich.
[13] A. S. Lange, "Automatic control of three-dimensional vector quantities—Part 1," IRE Trans. on Automatic Control, vol. AC-4, pp. 21–30; May, 1959.
[14] W. M. Cady, M. B. Karelitz, and L. A. Turner, "Radar Scanners and Radomes," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 26 p. 106, Fig. 4.1; 1948.
[15] H. Goldstein, "Classical Mechanics," Addison-Wesley Co., Reading, Mass., pp. 124–134; 1950.

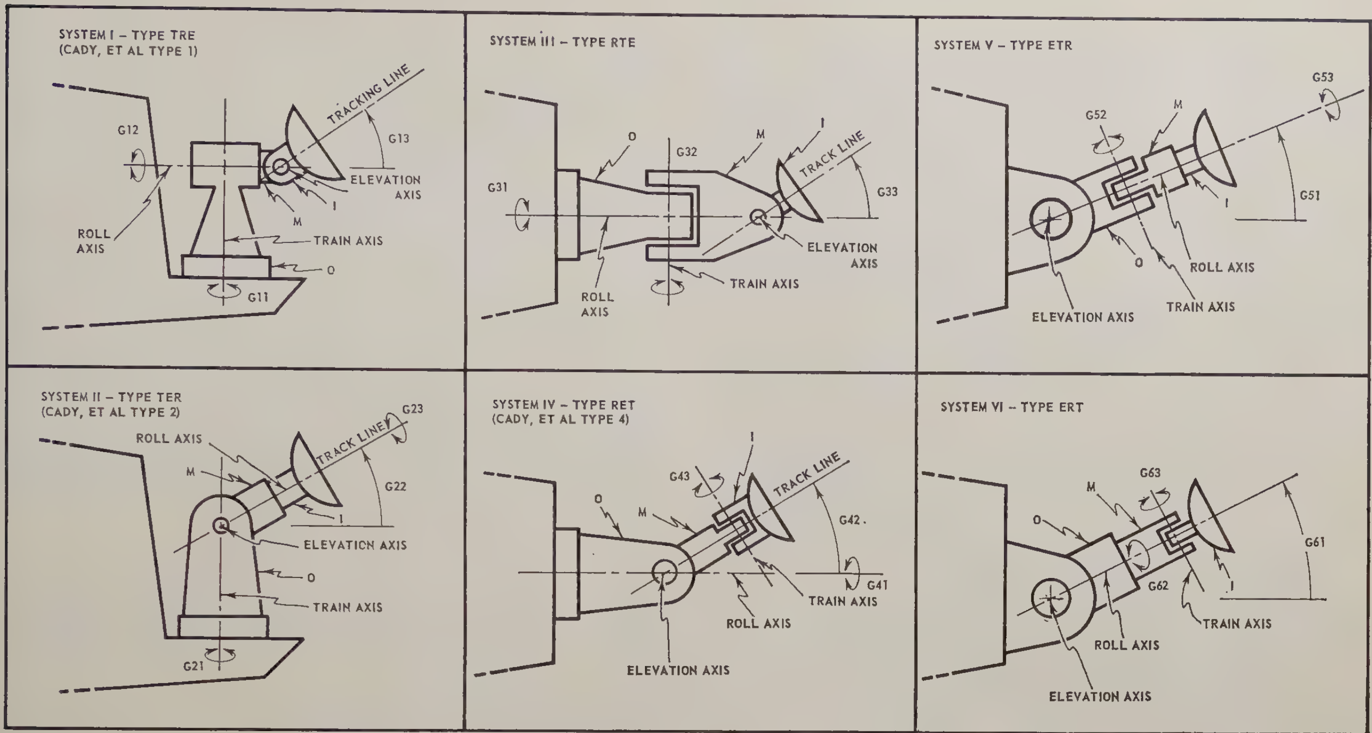


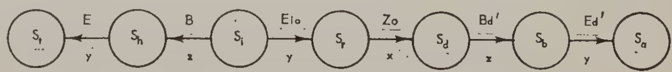
Fig. 6—Types of three-axis mounts. Key to symbols: Gii=gimbal angle; i indicates system $i=1, 2, 3 \cdots 6$, i indicates location (from base out) $i=1, 2, 3$. 1: between base and outer gimbal (O). 2: between outer gimbal (O) and middle gimbal (M). 3: between middle gimbal (M) and inner gimbal (I).

quired is considered in detail, and the methods used for determining the other servo parameters are considered only briefly.

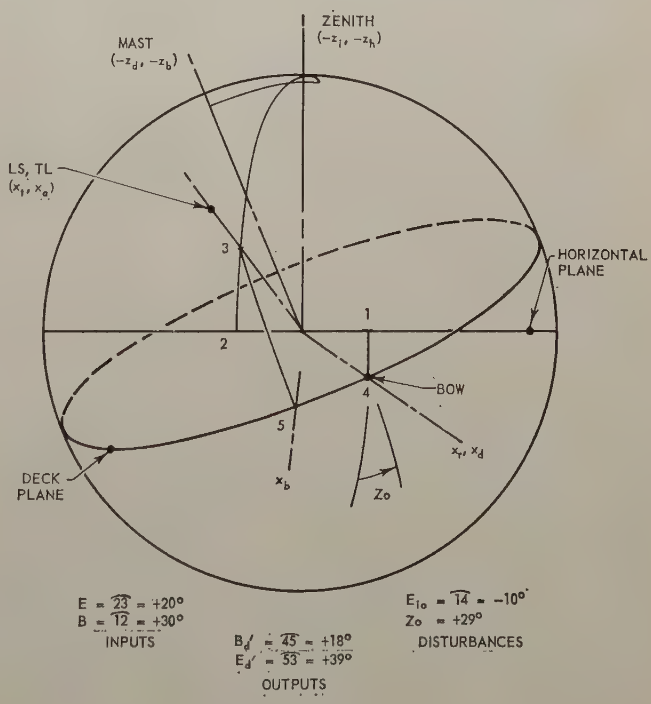
VI. Stabilizing the Two-Axis Mount[16]

Consider the two axis mount discussed in Section III of Part 1. It has been stated that two-axis mounts are used to isolate a controlled line from platform motion, and we consider here some details concerning the operation of such a mount. Fig. 2(a) of Part 1 illustrates the space-flow diagram for this mount, and Fig. 2(b) is a three-dimensional sketch of the angles which describe the problem; the definition of these angles and the coordinate sets associated with them given in Part 1 is repeated here.[17]

Let the target be located in space by the angles E and B and range, where B (bearing angle) is measured in the horizontal plane from the bow of the platform, and E (elevation angle) is measured about an axis in the horizontal plane. Let the instantaneous attitude of the platform be given by Eio (pitch angle) and Zo (roll angle). Eio is measured about an axis in the horizontal plane, and Zo is measured about the longitudinal axis of the platform. Further, let the telescope or antenna be positioned with respect to the platform by two angles, Bd' (antenna train angle) and Ed' (antenna elevation angle),



(a)



(b)

Fig. 2 (Part 1)—(a) Space-flow diagram. (b) Coordinate converter geometry.

[16] Cady, *et al., op. cit.*, Fig. 4.1(b).

[17] The auxiliary sets of coordinates located between S_i and S_d are of no interest here, since we assume that the motion of S_d with respect to S_i can be specified without regard to how these sets are defined. This topic is discussed in more detail in Appendix II.

where Bd' is about an axis normal to the deck of the platform and Ed' is about an axis in the deck of the platform. Fig. 2(a) is the space-flow diagram which describes this problem, and Fig. 2(b) is a three-dimensional sketch of the problem. Fig. 2(a) contains virtually all the information presented in Fig. 2(b), and at the same time, serves to indicate the flow of information in a way which is useful to the design of the coordinate converter. The S's in Fig. 2(a) represent sets of Cartesian coordinates which may be defined as follows:

- S_i = a reference set of coordinates, with the $(xy)_i$ plane horizontal, and the z_i axis directed vertically down.
- S_h = an auxiliary set of coordinates which results from a rotation B about z_i; $z_i \equiv z_h$ and the plane $(xy)_i$ is coincident with the plane $(xy)_h$. The $(xz)_h$ plane, like the $(xz)_i$ plane, is vertical, and contains $\overline{R}_{LS}$ and hence, the target.
- S_t = target coordinates, resulting from a rotation through the angle E about y_h so that $y_h \equiv y_t$; x_t is directed toward the target, so that $\overline{R}_{LS}$ is along x_t, and z_t forms an orthogonal set.
- S_r = an auxiliary set which results from the rotation through the angle Eio about $y_i \equiv y_r$; x_r is the longitudinal axis of the platform. The $(xz)_r$ plane is vertical and y_r is horizontal.
- S_d = platform coordinates, which result from the rotation Zo about x_r so that $(xy)_d$ defines the deck of the platform; if the platform is thought of as a ship, then the mast is directed along the negative z_d axis.
- S_b = a set of coordinates fixed to the outer gimbal, or train member. S_b results from the rotation Bd' about z_d so that $z_d \equiv z_b$, and the $(xy)_b$ plane is coincident with the $(xy)_d$ plane.
- S_a = a set of coordinates fixed to the inner gimbal or elevation member, resulting from the rotation Ed' about the y_b axis so that $y_b \equiv y_a$. The vector $\overline{R}_{TL}$ is directed along the x_a axis, and is therefore contained in the $(xz)_a$ plane.

It is customary to consider S_i as the Newtonian frame with respect to which the gyroscopic instruments measure motion. A Newtonian frame is usually defined,[18] in essence, as a frame in which Newton's second law is applicable. The physical significance of this somewhat circular definition is that the selection of a Newtonian frame depends on the particular problem at hand. For example, in the present problem of isolating some controlled member from the motion of the base on which it is mounted, it is usually sufficient to consider that the reference frame is moving with the uniform translational motion of the platform, but not participating with the platform's angular motion. However, if the problem is to design an inertial navigation system which must indicate position very accurately for many hours, it is necessary to use, as a Newtonian frame, a set of coordinates centered at the earth, but not rotating with it. Presumably, in order to design an inertial system for extraterrestrial navigation, it would be necessary to use a Newtonian frame defined by the fixed stars. The point here is that there are inaccuracies introduced by the choice of the Newtonian frame which may be made arbitrarily small by the proper choice of a Newtonian frame. When stabilizing a gun, for example, the errors introduced by neglecting the rotation of the earth or the translational motion of the platform are essentially undetectable.

It was previously stated that the requirement that the controlled member be isolated from the motion of the base may be written vectorially as

$$\overline{W}_{da} = -\overline{W}_{id}.$$

But, from Fig. 2(a), as well as the definitions listed above, it can be seen that $\overline{W}_{da} = \overline{W}_{db} + \overline{W}_{ba}$. Further, if this vector equation is to be useful for design purposes, it must be expressed in terms of components which represent measurable quantities. Therefore, (29) may be written as

$$\overline{W}^a_{db} + \overline{W}^a_{ba} = -\overline{W}^a_{id}, \tag{31}$$

where the superscript a indicates that the vectors are to be expressed in their S_a components. The three vectors in (31) will each be examined in detail. Consider first the term $\overline{W}_{db}$:

$$\overline{W}^a_{db} = T_{ab}\overline{W}^b_{db} = \begin{vmatrix} \cos Ed' & 0 & -\sin Ed' \\ 0 & 1 & 0 \\ \sin Ed' & 0 & \cos Ed' \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ DBd' \end{vmatrix} \tag{32}$$

where $D(\ )$[19] is the differential operator $d/dt(\ )$. T_{ab} is the transformation matrix which operates on a vector expressed in S_b components in order to express that vector in its S_a components. Since, as can be seen by Fig. 2(a), Bd' is the angle between S_b and S_d, and since this angle represents a rotation about the z_b, z_d axis, it follows that $|\overline{W}_{db}|$, the magnitude of $\overline{W}_{db}$, is equal to DBd', and that the direction of $\overline{W}_{db}$ is along the z_b axis.

Likewise, the second term in (31) may be expressed as

$$\overline{W}^a_{ba} = \begin{vmatrix} 0 \\ DEd' \\ 0 \end{vmatrix}; \tag{33}$$

i.e., the angular velocity of S_a with respect to S_b is equal to the time rate of change of the angle between them. As can be seen from Fig. 2(a), this angle is Ed', so $|\overline{W}_{ba}| = DEd'$, and since the rotation Ed' occurs about the y_a axis the direction of $\overline{W}_{ba}$ is along that axis.

[18] J. L. Synge and B. A. Griffith, "Principles of Mechanics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 32; 1959.

[19] Following "Standard Fire Control Symbols," Dept. of the Navy, Washington, D. C. BuOrd Publication No. OP-1700; 1950.

Finally, the last term in (31) may be written as follows:

$$\overline{W}_{id}^{a} = T_{ab}T_{bd}\overline{W}_{id}^{d} = \begin{vmatrix} \cos Ed' & 0 & -\sin Ed' \\ 0 & 1 & 0 \\ \sin Ed' & 0 & \cos Ed' \end{vmatrix} \cdot \begin{vmatrix} \cos Bd' & \sin Bd' & 0 \\ -\sin Bd' & \cos Bd' & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} w_x \\ w_y \\ w_z \end{vmatrix} \tag{34}$$

where T_{bd} is the transformation matrix required to transform a vector expressed in S_d components into its S_b components, and w_x, w_y, and w_z are the x, y, and z components, respectively, of $\overline{W}_{id}$ in S_d. (The several ways in which these components may be expressed are discussed in Appendix II.)

It is worthwhile to observe that (34) is more easily evaluated if the operations are performed from right to left. That is, first evaluate

$$\overline{W}_{id}^{b} = T_{bd}\overline{W}_{id}^{d} = \begin{vmatrix} w_x \cos Bd' + w_y \sin Bd' \\ -w_x \sin Bd' + w_y \cos Bd' \\ w_z \end{vmatrix}$$

and then

$$\overline{W}_{id}^{a} = T_{ab}\overline{W}_{id}^{b}$$

$$= \begin{vmatrix} (w_x \cos Bd' + w_y \sin Bd') \cos Ed' - w_z \sin Ed' \\ -w_x \sin Bd' + w_y \cos Bd' \\ (w_x \cos Bd' + w_y \sin Bd') \sin Ed' + w_z \cos Ed' \end{vmatrix}. \tag{35}$$

In contrast, if (34) is expanded from left to right, we have, first,

$$T_{ab} = T_{ab}T_{bd}$$

$$= \begin{vmatrix} \cos Ed' \cos Bd' & \cos Ed' \sin Bd' & -\sin Ed' \\ -\sin Bd' & \cos Bd' & 0 \\ \sin Ed' \cos Bd' & \sin Ed' \sin Bd' & \cos Bd' \end{vmatrix}$$

and secondly,

$$\overline{W}_{id}^{a} = T_{ab}\overline{W}_{id}^{d}$$

$$= \begin{vmatrix} w_x \cos Ed' \cos Bd' + w_y \cos Ed' \sin Bd' - w_z \sin Ed' \\ -w_x \sin Bd' + w_y \cos Bd' \\ w_x \sin Ed' \cos Bd' + w_y \sin Ed' \sin Bd' + w_z \cos Ed' \end{vmatrix}. \tag{36}$$

Of course, (35) and (36) are identical, but the steps followed in obtaining (35) are simpler because the result of each operation is a 3-element column matrix, whereas the first step in obtaining results is a 3×3 square matrix. Such matrices become more and more cumbersome with each successive operation, so that the right-to-left sequence is even more important when a number of transformations are to be performed.

When the indicated operations are performed, (32)–(34) can be substituted into (31), giving

$$-\begin{vmatrix} -DBd' \sin Ed' \\ DEd' \\ DBd' \cos Ed' \end{vmatrix}$$

$$= \begin{vmatrix} (w_x \cos Bd' + w_y \sin Bd') \cos Ed' - w_z \sin Ed' \\ -w_x \sin Bd' + w_y \cos Bd' \\ (w_x \cos Bd' + w_y \sin Bd') \sin Ed' + w_z \cos Ed' \end{vmatrix} \tag{37}$$

so that

$$DEd' = w_x \sin Bd' - w_y \cos Bd', \tag{38}$$

$$DBd' = -\left[w_z + (w_x \cos Bd' + w_y \sin Bd') \tan Ed'\right]. \tag{39}$$

Eqs. (38) and (39) express the outputs of the elevation and train servos, respectively, which are required to isolate the controlled member from the motion of the base. It may be observed that $\overline{W}_{ia}$ is not exactly zero; *i.e.*,

$$\overline{W}_{ia}^{a} = \begin{vmatrix} (w_x \cos Bd' + w_y \sin Bd') \sec Ed' \\ 0 \\ 0 \end{vmatrix}. \tag{40}$$

However, if x_a (the x axis of the coordinate set S_a) is the barrel of a gun, or the tracking line of a radar antenna, or a telescope, then the controlled line itself is stationary with respect to S_i, although the gun or the antenna (the controlled member) is rotating, with respect to S_i, about the controlled line. In many instances, this rotation is not important. Note that the vector equation $\overline{W}_{ia}=0$ expresses three conditions which must be satisfied. However, the two-gimbal mount supplies only two constraints, corresponding to the outputs of the train and elevation servos. If it is desired to satisfy (31) identically, it is necessary to introduce a third constraint, in the form of another gimbal. A three-gimbal mount which satisfies (31) is discussed in the following section. It will suffice here to observe that to stabilize a *line*, two gimbals are required, and to stabilize a *plane*, three gimbals are required. (It is presumed in this statement that there is no redundancy in the gimbal system. Redundancy has been considered in Section IV of Part 1 and is discussed further below).

It is necessary, next, to examine in detail how a mechanism capable of solving (38) and (39) can be instrumented. That is, (38) and (39) represent the values of DEd' and DBd' required to stabilize the tracking line, but these are the outputs of the elevation and train servos. What are their inputs, and how are the servo loops closed? Two methods using gyroscopic instruments are common: 1) measure the motion of the platform with respect to inertial space, or 2) measure the motion of the tracking line with respect to inertial space. The electrical signals from the gyroscopic instruments form the inputs to the servos, and the motion of the tracking line, in one case with respect to the platform,

and, in the other, with respect to inertial space, act to close the servo loops. One of these methods is called "director stabilization," and the other is called "on-mount stabilization." In the director system, a vertical gyro (or, in general, a two-gimbal gyro) is used to measure the motion of the base with respect to inertial space. The signals from the gyro are modified by means of a coordinate converter (see Part 1, Section III) to obtain the servo inputs. In fact, as was pointed out, if the servos which drive the BD' and Ed' resolvers in Fig. 3(b) of Part I are also used to position the controlled member, the controlled line is isolated from the motion of the base; it can be seen that (38) and (39) can be obtained by differentiating (21) and (17) of Part I, respectively. The name director stabilization is derived from the fact that the vertical gyro and the coordinate converter are usually located in the fire control director.

The name on-mount stabilization, on the other hand, is derived from the fact that the gyros are mounted directly on the controlled member. Single gimbal, or rate gyros have been widely used for this purpose,[20] and this is the configuration which will be assumed in the following discussion. Fig. 7 is a servo block diagram of one axis of the base motion isolation system utilizing on-mount stabilization. Note that the servo loop is closed by means of the motion of the controlled member with respect to inertial space, as measured by the gyroscope.

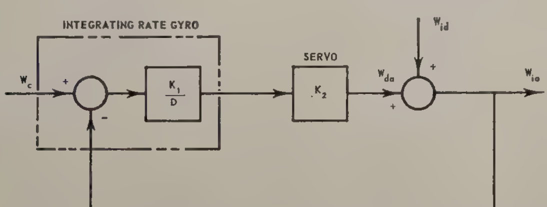


Fig. 7—Servo block diagram for stabilization loop (single axis). $W_c = W_{ia'}$ commanded, W_{ia} = angular rate of S_a with respect to S_i, W_{id} = angular rate of S_d with respect to S_i, W_{da} = angular rate of S_a with respect to S_d.

The rate gyros (a detailed discussion of the operation of the rate gyro will be given in Part 3) mounted on the controlled member measure the y and z components of $\overline{W}_{ia}$. (In this case, the x component is not measured for base motion isolation purposes because there is no way to constrain it to zero.) The outputs of the y and z gyros can be determined from the expression for $\overline{W}_{ia}^a$, where

$$\overline{W}_{ia}^a = \overline{W}_{id}^a + \overline{W}_{db}^a + \overline{W}_{ba}^a.$$

Then, from (32)–(34)

$$(\overline{W}_{ia})_y^a = DEd' - w_x \sin Bd' + w_y \cos Bd' \qquad (41)$$

$$(W_{ia})_z^a = (DBd' + w_z) \cos Ed' + (w_x \cos Bd' + w_y \sin Bd') \sin Ed' \qquad (42)$$

where $(W_{ia})_y^a$ is the y component of $\overline{W}_{ia}^a$, measured by the y-axis gyro, and $(W_{ia})_z^a$ is the z component of $\overline{W}_{ia}^a$, measured by the z-axis gyro.

Consider first the elevation servo; from the block diagram shown in Fig. 7, the output of the elevation servo is

$$DEd' = \frac{(K_1K_2)_e}{D}\left[W_{ce} - (W_{ia})_y^a\right] \qquad (43)$$

where K_1 is the gyro sensitivity, K_2 is the servo gain, and and the subscript e designates the elevation servo.

Now, if W_{ce} (the commanded elevation rate) is zero, then (43) becomes

$$DEd' = \frac{1}{\dfrac{D}{(K_1K_2)_e} + 1}(w_x \sin Bd' - w_y \cos Bd') \qquad (44)$$

and for $(K_1K_2)e_i \gg D$, (44) may be approximated by

$$DEd' \approx (w_x \sin Bd' - w_y \cos Bd'). \qquad (45)$$

In a similar fashion, for the train servo,

$$DBd' = \frac{(K_1K_2)_t}{D}\left[W_{ct} - (W_{ia})_z^a\right] \qquad (46)$$

where the subscript t designates the train servo.

For $W_{ct} = 0$,

$$DBd' = -\frac{1}{\dfrac{D \sec Ed'}{(K_1K_2)_t} + 1}\left[w_z + (w_x \cos Bd + w_y \sin Bd') \tan Ed'\right]. \qquad (47)$$

As before, if $(K_1K_2)_t \cos Ed' \gg D$, (47) becomes

$$-DBd' \approx \left[w_z + (w_x \cos Bd' + w_y \sin Bd') \tan Ed'\right]. \qquad (48)$$

Since (45) ≈ (38) and (48) ≈ (36), it can be seen that the train and elevation servos do, in fact, stabilize the controlled line when they are suitably connected to the y and z gyro outputs, respectively.

It may be observed from (47) that the time constant of the train servo varies inversely with cosine of the elevation angle Ed'. This is because the input axis of the z-axis gyro is misaligned with the train servo output axis by the angle Ed'. As a consequence, the loop gain is low for large values of Ed' (going to zero as $Ed' \rightarrow \pi/2$) and the performance of the train servo varies with Ed'. If Ed' varies over a significant range of values, it is customary to insert an automatic gain control (AGC) device to eliminate this effect. The automatic gain control may be accomplished by multiplying the output of the z-axis gyro by sec Ed'. This may be done with a resolver (see Appendix I) but usually it is sufficient to use a non-linear potentiometer, wound so as to modify the applied voltage by the secant of the potentiometer shaft angle.

[20] C. S. Draper, *et al.*, "The Floating Integrating Gyro, and its Application to Geometrical Stabilization Problems on Moving Bases," Inst. Aeronaut. Sciences, New York, N. Y., Rept. No. 503; January, 1955.

If the shaft is driven by Ed', (46) becomes

$$DBd' = \frac{(K_1K_2)t}{D}\left[W_{et} - (W_{ia})_z^a\right] \sec Ed' \tag{49}$$

from which it follows that

$$DBd' = -\frac{1}{\dfrac{D}{(K_1K_2)_t} + 1}\left[w_z + (w_x \cos Bd' + w_y \sin Bd')\right]; \tag{50}$$

i.e., the variation of the train servo time constant with Ed' has been eliminated. It must be emphasized that the purpose of this operation is to maintain a constant loop gain for wide variations in Ed'. This AGC function is not so clear in three axis mounts, so it is important that this function be clearly understood in this more simple example. The function sec Ed' is difficult to mechanize as $Ed' \to \pi/2$, since sec $\pi/2 \to \infty$. However, it can be seen that for $Ed' \to \pi/2$, this two-axis mount exhibits a trigonometric singularity which means that operation for Ed' near $\pi/2$ must be avoided. Therefore, the AGC operation will be performed satisfactorily for all values of Ed' for which the system may be expected to operate.

VII. A Three-Axis Stable Platform

It was shown in the previous section that a two-axis gimbal arrangement, although geometrically stabilizing a controlled line, allowed rotation of the controlled member about the controlled line, where, it may be recalled, the controlled line is an axis fixed to the controlled member. It is of interest, therefore, to consider a three-axis gimbal arrangement, which has three conditions of restraint, one of which can be used to restrain $(W_{ia})_x{}^a$ to zero, thus eliminating the rotation of the controlled member about the controlled line.

As an example, we will take for our model the cross-level gimbal arrangement[21] shown as System I of Fig. 6. The "cross-level" arrangement is similar to the train-elevation system described in the previous section, except that an extra gimbal has been added between the train and the elevation gimbals. The function of this extra gimbal is to maintain the elevation axis level, or horizontal. The name cross-level refers to the extra axis, which is oriented 90° with the elevation axis; hence it crosses with the level axis. Notice that the elevation axis may be thought of as the controlled line of a two-axis mount; hence, it can be demonstrated by arguments similar to those used in Section VI that the controlled *line* (the elevation axis) is isolated from the motion of the base. It follows, therefore, that if a controlled *member* is mounted on this axis, the controlled *member* can be completely isolated from the motion of the base, since it can rotate about the controlled line in such a way as to remain fixed in space.

[21] Cady, *et al.*, *op. cit.*, Fig. 4.

The space-flow diagram which represents this gimbal configuration is shown in Fig. 8. The S's in Fig. 8 represent sets of Cartesian coordinates which are defined as follows.

S_i = a reference set of coordinates, with the $(xy)_i$ plane horizontal, and the z_i axis directed vertically down.

S_d = platform coordinates, with $(xy)_d$ defining the deck of the platform.

S_b = a set of coordinates fixed to the outer gimbal or train member. S_b results from the rotation Bd about z_d so that $z_d = z_b$ and the $(xy)_b$ plane is coincident with the $(xy)_d$ plane.

S_c = a set of coordinates fixed to the middle or cross-level gimbal. S_c results from the rotation Zd about the x_b axis so that $x_b = x_c$. The $(xy)_c$ plane is horizontal.

S_a = a set of coordinates fixed to the inner or elevation gimbal. S_a defines the controlled member, and results from the rotation Ed about the y_c axis so that $y_c = y_a$. Since the $(xy)_c$ plane is horizontal the $(xz)_a$ plane is vertical.

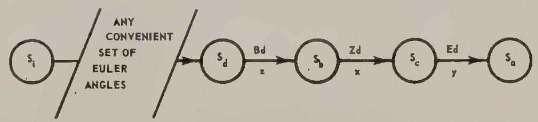


Fig. 8—Space-flow diagram cross-level system. (See Appendix II.)

Bd and Ed are the train and elevation angles, respectively, analogous to Bd' and Ed'. (In general, primed quantities are used in OP-1700[19] to represent "unstabilized" quantities, as distinguished from "stabilized" quantities which are so named because the angle Ed lies in the vertical plane $(xz)_a$, which remains fixed with respect to the reference space when directed at a stationary target.)

The requirement that the controlled member be isolated from the motion of the base is given by the following vector equation:

$$\overset{a}{W}_{ia} = 0. \tag{51a}$$

But, from Fig. 8, $\overline{W}_{ia} = \overline{W}_{id} + \overline{W}_{db} + \overline{W}_{bc} + \overline{W}_{ca}$ so that (51a) may be written

$$\overset{a}{\overline{W}}_{db} + \overset{a}{\overline{W}}_{bc} + \overset{a}{\overline{W}}_{ca} = -\overset{a}{\overline{W}}_{id} \tag{51b}$$

where the superscript a indicates that the vectors are to be expressed in S_a coordinates, and

$\overline{W}_{id}$ = angular rate of S_d with respect to S_i,

$\overline{W}_{db}$ = angular rate of S_b with respect to S_d,

$\overline{W}_{bc}$ = angular rate of S_c with respect to S_b,

$\overline{W}_{ca}$ = angular rate of S_a with respect to S_c.

The four vectors in (51b) will each be examined as follows:

$$\overline{W}_{db}^{a} = T_{ac}T_{cb}\overline{W}_{db}^{b} = \begin{vmatrix} \cos Ed & 0 & -\sin Ed \\ 0 & 1 & 0 \\ \sin Ed & 0 & \cos Ed \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zd & \sin Zd \\ 0 & -\sin Zd & \cos Zd \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ DBd \end{vmatrix} \tag{52}$$

T_{ac} is the transformation matrix which operates on a vector expressed in S_c in order to express it in S_a, and T_{cb} is the matrix transforming components from S_b to S_c. Since Bd is the angle between S_d and S_b, $|\overline{W}_{db}|$, the magnitude of W_{db}, is equal to DBd, and since the common axis between S_d and S_b is the z axis, it is directed along this axis.

$$\overline{W}_{bc}^{a} = T_{ac}\overline{W}_{bc}^{c} = \begin{vmatrix} \cos Ed & 0 & -\sin Ed \\ 0 & 1 & 0 \\ \sin Ed & 0 & \cos Ed \end{vmatrix} \begin{vmatrix} DZd \\ 0 \\ 0 \end{vmatrix} \tag{53}$$

Zd is the angle between S_b and S_c, with $x_b = x_c$. Therefore, $|\overline{W}_{bc}| = DZd$, and $\overline{W}_{bc}$ is directed along the x direction in both S_b and S_c.

$$\overline{W}_{ac}^{a} = \begin{vmatrix} 0 \\ DEd \\ 0 \end{vmatrix}. \tag{54}$$

Fig. 8 shows that the angle between S_c and S_a is Ed, and the axis of rotation is their common y axis. Therefore, the rate of change of Ed is given by DEd, directed along y_a.

$$\overline{W}_{id}^{a} = T_{ac}T_{cb}T_{bd}\overline{W}_{id}^{d}$$

$$= \begin{vmatrix} \cos Ed & 0 & -\sin Ed \\ 0 & 1 & 0 \\ \sin Ed & 0 & \cos Ed \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zd & \sin Zd \\ 0 & -\sin Zd & \cos Zd \end{vmatrix} \begin{vmatrix} \cos Bd & \sin Bd & 0 \\ -\sin Bd & \cos Bd & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} w_x \\ w_y \\ w_z \end{vmatrix} \tag{55}$$

where T_{bd} transforms the S_d components of $\overline{W}_{id}$ into S_b components. Note that in Fig. 8, no particular set of angles between S_i and S_d is specified. $\overline{W}_{id}$ represents a measurable physical quantity, and the representation shown in Fig. 8 is intended to emphasize that the vector $\overline{W}_{id}$ is completely independent of the various ways it can be described mathematically. It is customary to speak of the three angles relating the attitude of an object with respect to some reference coordinates as "Eulerian angles."[22] Expressions for $\overline{W}_{id}$ in terms of Eulerian angles are discussed in more detail in Appendix II.

[22] Goldstein, *op. cit.*, pp. 107–109.

Eq. (51) can be rewritten by combining (52)–(55), giving

$$\begin{vmatrix} DZd\cos Ed - DBd\cos Zd\sin Ed \\ DEd + DBd\sin Zd \\ DZd\sin Ed + DBd\cos Zd\cos Ed \end{vmatrix} = -\begin{vmatrix} X\cos Ed - w_z\cos Zd\sin Ed + Y\sin Zd\sin Ed \\ Y\cos Zd + W_z\sin Zd \\ X\sin Ed + w_z\cos Zd\cos Ed - Y\sin Zd\cos Ed \end{vmatrix} \tag{51c}$$

where

$$X = (w_x\cos Bd + w_y\sin Bd),$$
$$Y = -(w_x\sin Bd - w_y\cos Bd).$$

The following scalar equations may be obtained from (51c):

$$DZd\cos Ed - DBd\cos Zd\sin Ed = -[-X\cos Ed + Y\sin Zd\sin Ed - w_z\cos Zd\sin Ed] \tag{56}$$

$$DEd + BDd\sin Zd = -[Y\cos Zd + w_z\sin Zd], \tag{57}$$

$$DZd\sin Ed + DBd\cos Zd\cos Ed = -[X\sin Ed - Y\sin Zd\cos Ed + w_z\cos Zd\cos Ed] \tag{58}$$

From these three equations we seek expressions for DZd, DBd, and DEd, which are the outputs of the servos which drive the three gimbals. These expressions may be found by the following steps.

1) The sum of cos Ed (56) and sin Ed (58) gives

$$DZd = -X = -(w_x\cos Bd + w_y\sin Bd). \tag{59}$$

2) The sum of—sin Ed (56) and cos Ed (58) gives

$$DBd = -w_z + Y\tan Zd$$
$$= -[w_z + (w_x\sin Bd - w_y\cos Bd)\tan Zd]. \tag{60}$$

3) Finally, the substitution of (60) into (57) gives

$$DEd = \sec Zd(w_x\sin Bd - w_y\cos Bd). \tag{61}$$

Eqs. (59)–(61) express the outputs of the three gimbal servos, for the middle gimbal, outer gimbal and inner gimbal, respectively, which are required to make $\overline{W}_{ia} = 0$. The work of deriving these expressions may be simplified somewhat by recalling that the function of the cross-level servo drive, whose output is DZd, is to maintain the elevation axis y_c, horizontal. Mathematically, this may be expressed as $(W_{ic})_x^c = (W_{ic})_x^b = 0$. That is, if the cross-level gimbal is not rotating about its x axis with respect to the reference frame, then its y axis is not rotating with respect to the reference frame. Therefore, if we express $\overline{W}_{ic}^{b}$ as

$$\overline{W}_{ic}^{b} = T_{bd}\overline{W}_{id}^{d} + \overline{W}_{db}^{b} + T_{bc}\overline{W}_{bc}^{c}$$

$$= \begin{vmatrix} w_x \cos Bd + w_y \sin Bd \\ -w_x \sin Bd + w_y \cos Bd \\ w_z \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ DBd \end{vmatrix} + \begin{vmatrix} DZd \\ 0 \\ 0 \end{vmatrix} \qquad (62)$$

then the condition that $(W_{ic})_x^b = 0$ gives

$$DZd = -(w_x \cos Bd + w_y \sin Bd) \qquad (63)$$

and (62) reduces to

$$\overline{W}_{ic}^{b} = \begin{vmatrix} 0 \\ -w_x \sin Bd + w_y \cos Bd \\ DBd + w_z \end{vmatrix}. \qquad (64)$$

The vector equation specifying that the controlled member S_a be irrotational with respect to the reference frame is given by

$$\overline{W}_{ia}^{a} = T_{ac}T_{cb}\overline{W}_{ic}^{b} + \overline{W}_{ca}^{a} = 0$$

which may be expressed in terms of the now simplified $\overline{W}_{ic}$ to give

$$\begin{vmatrix} Y \sin Zd \sin Ed - (BDd + w_z) \cos Zd \sin Ed \\ Y \cos Zd + (DBd + w_z) + \sin Zd + DEd \\ -Y \sin Zd \cos Ed + (DBd + w_z) \cos Zd \cos Ed \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}. \qquad (65)$$

Either the x or the z component of (65) can be used to give the expression for DBd as

$$DBd = -[w_z + (w_x \sin Bd - w_y \cos Bd) \tan Zd]. \qquad (66)$$

Eq. (66) may be substituted into the y component of (65), to give

$$DEd = \sec Zd(w_x \sin Bd - w_y \cos Bd). \qquad (67)$$

Eq. (63), (66), and (67) are of course identical to (59)–(61), respectively. However, it may be noted that the second method of derivation is more simple and requires fewer algebraic operations; also, this second method demonstrates the "levelling" operation of the middle gimbal servo.

The discussion thus far has been concerned with the determination of the gimbal servo rates required to isolate the controlled member S_a from the motion of the base S_d. The next step is to determine how the servos must be arranged to accomplish this base motion isolation. From (51c), the outputs of x, y, and z gyros mounted on S_a may be inferred as

$$(W_{ia})_x^a = DZd \cos Ed - DBd \cos Zd \sin Ed + X \cos Ed + Y \sin Zd \sin Ed - w_z \cos Zd \sin Ed \qquad (68)$$

$$(W_{ia})_y^a = DEd + DBd \sin Zd + Y \cos Zd + w_z \sin Zd \qquad (69)$$

$$(W_{ia})_z^a = DZd \sin Ed + DBd \cos Zd \cos Ed + X \sin Ed - Y \sin Zd \cos Ed + w_z \cos Zd \cos Ed. \qquad (70)$$

The servo output quantity is equal to K_1K_2/D times the difference between the commanded rate and the actual rate measured by the gyro, where K_1 is the gyro sensitivity, and K_2 is the servo gain, as is shown in Fig. 7. Therefore, if the subscripts I, M, and O are used to designate the inner, middle, and outer gimbal servos, respectively, we have

$$DZd = \frac{(K_1K_2)_M}{D}[W_{xc} - (W_{ia})_x^a] \qquad (71)$$

$$DEd = \frac{(K_1K_2)_I}{D}[W_{yc} - (W_{ia})_y^a] \qquad (72)$$

$$DBd = \frac{(K_1K_2)_O}{D}[W_{zc} - (W_{ia}^a)_z] \qquad (73)$$

where, as before, $D = d/dt$ ( ).

Now, for W_{xc}, W_{yc}, and W_{zc}, the x, y, and z commands, respectively, all equal to zero, we have

$$DZd = \frac{1}{\dfrac{D \sec Ed}{(K_1K_2)_M} + 1}[(DBd + w_z) \cos Zd \tan Ed - X - Y \sin Zd \tan Ed] \qquad (74a)$$

$$DEd = \frac{1}{\dfrac{D}{(K_1K_2)_I} + 1}[-(DBd + w_z) \sin Z_d - Y \cos Zd] \qquad (75a)$$

$$DBd = \frac{1}{\dfrac{D \sec Zd \sec Ed}{(K_1K_2)_O} + 1}[-w_z + Y \tan Zd - (DZd + X) \sec Zd \tan Ed]. \qquad (76a)$$

It can be observed that the servo loop time constant for the cross-level and train servos, (74) and (76), respectively, varies with the angle Ed, for the former, and with the angles Ed and Zd, for the latter. This variation in the loop gain can be compensated for in the same manner described in Section VIII, by the use of secant potentiometers for the automatic gain control operation. However, before discussing the AGC operator requirements, it is of interest to first demonstrate that, in theory at least, the gimbal drives function properly even without the AGC feature.

Assume that

$$(K_1K_2)_M \gg D \sec Ed$$
$$(K_1K_2)_I = D$$
$$(K_1K_2)_O = D \sec Ed \sec Zd;$$

i.e., assume that

$$\frac{\sec Ed}{(K_1K_2)_I} D^2Zd$$

is much smaller that DZd and

$$\frac{\sec Ed \sec Zd}{(K_1K_2)_O} D^2Bd$$

is much smaller than DBd. If the system is stable, so that DBd and DZd reach some equilibrium value, then this is a valid assumption, and our purpose in making this assumption is to examine the equilibrium behavior of the system. (The determination of stability is a more difficult problem; the consideration of stability in a simple example is considered in Appendix I). These assumptions are somewhat analogous to the application of the final value theorem of Laplace transformation theory.[23] Using these assumptions, (74a), (75a) and (76a) become, respectively,

$$DZd \approx [(DBd + w_z) \cos Zd \tan Ed - X - Y \sin Zd \tan Ed], \quad (74b)$$

$$DEd \approx [-(DBd + w_z) \sin Zd - Y \cos Zd], \quad (75b)$$

$$DBd \approx [-w_z + Y \tan Zd + (DZd + X) \sec Zd \tan Ed]. \quad (76b)$$

The sum of Equation (74b) and $\cos Zd \tan Ed$ (76b) gives

$$DZd = -X = -(w_x \cos Bd + w_y \sin Bd) \quad (77)$$

The substitution of (77) into either (74b) or (76b) gives

$$DBd = -w_z + Y \tan Zd = [-w_z + (w_x \sin Bd - w_y \cos Bd) \tan Zd] \quad (78)$$

and finally, the substitution of (78) into (75b) gives

$$DEd = \sec Zd (w_x \sin Bd - w_y \cos Bd). \quad (79)$$

Since (77)–(79) are identical to (59)–(61), respectively, it can be concluded that the fact that the gyro axes and the servo axes are not always aligned does not effect the steady-state performance of the system. However, the degeneration of the dynamic performance implicit (74a) and (76a) is not generally tolerable, so that it is necessary to compensate for the misalignment of the gyro and servo axes by multiplying the output of the gyro by the secant of the angles of misalignment. For example, (71) becomes

$$DZd = \frac{(K_1K_2)_M}{D} [W_{xc} - (W_{ia})_x^a] \sec Ed \quad (80a)$$

and for $W_{xc} = 0$, we have

$$DZd = \frac{1}{\dfrac{D}{(K_1K_2)_M} + 1} [(DBd + w_z) \cos Zd \tan Ed - X - Y \sin Zd \tan Ed]. \quad (80b)$$

Since the input axis of the y gryo is always aligned with the inner gimbal servo output axis, no AGC is required for the elevation servo. However, the train servo axis is misaligned by both the Ed and Zd angles, so that the AGC operator used to correct the z axis gyro has the form $\sec Ed \sec Zd$. Therefore (73) becomes

$$DBd = \frac{(K_1K_2)_O}{D} [W_{zc} - (W_{ia})_z^a] \sec Ed \sec Zd. \quad (81a)$$

For $W_{zc} = 0$ (53a) becomes

$$DBd = \frac{1}{\dfrac{D}{(K_1K_2)_O} + 1} [-w_z + Y \tan Zd - (DZd + X) \sec Zd \tan Ed]. \quad (81b)$$

It can be seen that the term in the brackets of (80b) is the same as the similar term in (74a); likewise, the bracketed term in (81b) is identical to the bracketed term in (76a). Therefore, it may be concluded that the AGC operator effects only the servo-loop time constant, and does not change the steady-state performance of the system.

In many applications, Ed and Zd are relatively small angles (*i.e.*, less than $\pi/2$). In such cases, it is convenient to use potentiometers to perform the AGC function, since potentiometers are less expensive, lighter and smaller than resolvers. In fact, since high accuracy is not required for this operation, linear potentiometers may be used.[24] However, if Ed or Zd perform larger excursions, the use of resolvers is required to adequately perform these AGC operations.

The location of the resolvers in the stabilization system may be determined by the following considerations. The three gyros measure the x, y and z components of $\overline{W}_{ia}^a$. The components of $\overline{W}_{ia}$ in Sc may be obtained by applying $(W_{ia})_x^a$ and $(W_{ia})_z^a$ to a resolver driven by Ed, since $(W_{ia})_y^a = (W_{ia})_y^c$. Then

$$\overline{W}_{ia}^c = T_{ca}\overline{W}_{ia}^a = \begin{vmatrix} \cos Ed & 0 & \sin Ed \\ 0 & 1 & 0 \\ -\sin Ed & 0 & \cos Ed \end{vmatrix} \begin{vmatrix} (W_{ia})_x^a \\ (W_{ia})_y^a \\ (W_{ia})_z^a \end{vmatrix} = \begin{vmatrix} (W_{ia})_x^a \cos Ed + (W_{ia})_z^a \sin Ed \\ (W_{ia})_y^a \\ -(W_{ia})_x^a \sin Ed + (W_{ia})_z^a \cos Ed \end{vmatrix} = \begin{vmatrix} (W_{ia})_x^c \\ (W_{ia})_y^c \\ (W_{ia})_z^c \end{vmatrix}$$

[23] M. F. Gardner and J. L. Barnes, "Transients in Linear Systems," John Wiley and Sons, Inc., New York, N. Y., vol. 1, pp. 265–267; 1942.

[24] J. A. Greenwood, Jr., J. V. Holdam, Jr., and D. MacRae, Jr., "Electronic Instruments," M.I.T. Rad. Lab. Ser., McGraw-Hill Book Co., Inc., New York, N. Y., vol. 21, pp. 102–103; 1948.

The x component of $\overline{W}_{ia}^c$ can be evaluated from (68) and (70) to give

$$(W_{ia})_x^c = DZd + X. \tag{82}$$

If $(W_{ia})_z^a$ is applied to the cross-roll servo, the servo output may be written as

$$DZd = \frac{(K_1K_2)_M}{D}(W_{xc} - DZd - X)$$

which reduces to the familiar form

$$DZd = \frac{W_{xc} - X}{\dfrac{D}{(K_1K_2)_M} + 1} \approx (W_{cx} - X), \tag{83}$$

and for $W_{cx}=0$, $DZd \approx -X$, which is the same result found earlier in (59), (63), and (77). Inspection of (83) shows that the Zd servo gain is independent of the value of Ed. Consequently we may conclude that the process of generating the Zd servo signal by passing the X- and Z-axis gyro signals through an Ed resolver is the correct operation. Since $DZd \approx -X$, it follows that $(W_{ia})_x^a = 0$.

The term $(W_{ia})_x^c$ can be evaluated from (67) and (69) to give

$$(W_{ia})_y^a = (DBd + w_z)\sin Zd - Y\frac{\sin^2 Zd}{\cos Zd}\cdot \tag{84}$$

Likewise, the z component of $\overline{W}_{ia}^c$ can be evaluated from (68) and (70) to give

$$(W_{ia})_z^c = (DBd + w_z)\cos Zd - Y\sin Zd. \tag{85}$$

Therefore,

$$W_{ia} = \begin{vmatrix} 0 \\ (DBd + w_z)\quad \sin Zd - Y\dfrac{\sin^2 Zd}{\cos Zd} \\ (DBd + w_z)\quad \cos Zd - Y\quad \sin Zd \end{vmatrix}. \tag{86}$$

The vector $\overline{W}_{ia}$ can be resolved into its S_b components as follows:

$$\overline{W}_{ia}^b = T_{bc}\overline{W}_{ia}^c = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zd & -\sin Zd \\ 0 & \sin Zd & \cos Zd \end{vmatrix} \begin{vmatrix} 0 \\ (W_{ia})_y^c \\ (W_{ia})_z^c \end{vmatrix}$$

$$= \begin{vmatrix} 0 \\ 0 \\ (DBd + w_z) - \tan Zd Y \end{vmatrix} \tag{87}$$

so that

$$DBd = \frac{(K_1K_2)_O}{D}[-(DBd + w_z) + \tan Zd Y]$$

or finally,

$$DBd \approx -w_z + Y\tan Zd \tag{88}$$

which is the desired result.

Fig. 9 shows the schematic arrangement of the gyros, resolvers, and servos required for these operations. The continuous lines show the signals and the dashed lines show mechanical connections. Note that the y-axis gyro signal goes to the y-axis servo and to the Zd resolver. The output of the y-axis servo is shown as driving both the three gyros and the Ed resolver, to which is applied the outputs of x- and z-axis gyros. One of the signals from this resolver is applied to the x-axis servo, which generates Zd. Zd drives the inner gimbal (via the dashed line to the y-axis servo, on the diagram) and the Zd resolver. The other signal from the Ed resolver, together with the y-axis gyro signal, is applied to the Zd resolver. One of the outputs of this resolver is zero; the other is applied to the Bd servo, which drives the middle gimbal. Platform motion is shown as driving the outer gimbal.

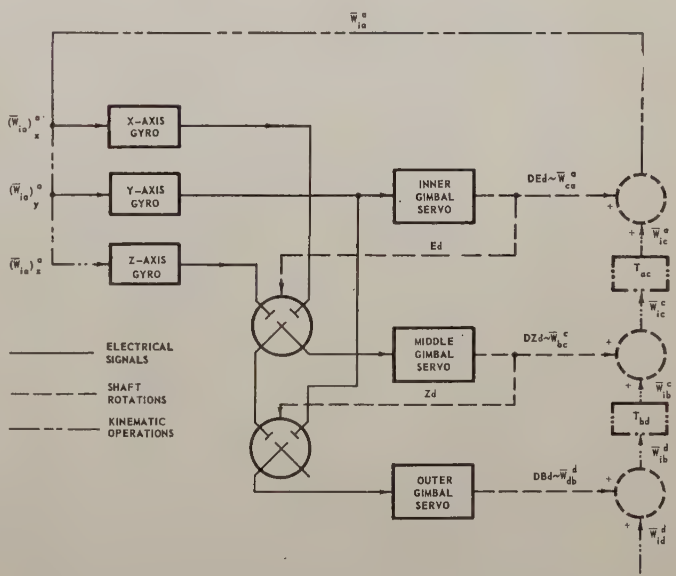


Fig. 9—Schematic diagram of three-axis stabilization system.

VIII. Design Considerations For Multiple-Gimbal Systems

The preceding sections have considered in some detail the analysis of two base-motion isolation systems. These analyses serve two purposes: one, to illustrate an analytic procedure which is useful in determining the dynamic relations which affect the performance of a complex automatic control system, and, two, to obtain quantitative information necessary for the rational design of such control systems. The purpose of this section is to briefly consider certain additional aspects of those multiple-gimbal design problems which are of interest to the automatic controls engineer.

In Section IV of Part 1, which was concerned with gimbal lock, the effect of trigonometric singularities was discussed in some detail. It was asserted there that these singularities or poles are associated with the problem of tracking a target near the extension in space of the outer gimbal axis. For example, consider that the controlled member is a radar antenna tracking a target; it was demonstrated in Section IV, that for small changes in either the input or the disturbance quantities, large changes in the output quantity are commanded, requiring unachievable angular rates and angular accelerations. As a consequence, the servo drives cannot keep the tracking error small enough to continue tracking, and the target is lost. Thus, it is said that there is a "hole" in the space coverage of the controlled member. Since the trigonometric singularities move with the craft, it is clear that the holes in the coverage of the antenna are determined by the orientation of the gimbal axes, the motion of the craft, and the servo rates.[25] The motion of the craft is generally beyond the control of engineer charged with the design of the stabilization system; in fact, it represents the operational environment of the system. Further, the designer is limited in the torques and speeds he can expect the servos to develop, so it is necessary to consider how these holes in the antenna coverage may be minimized or eliminated. One method of alleviating this difficulty is to align the outer axis so that it is directed through some spot where the antenna is already blind. Fig. 10 shows a sketch of a shipboard installation,[26] in which the poles associated with a radar antenna mount are made to coincide with the fantail and the superstructure of the ship. Another method of minimizing the holes in the antenna coverage is to add another axis, utilizing a four-axis mount instead of a three-axis mount. This redundancy can be employed to eliminate holes and reduce the servo rates required to stabilize the antenna, but creates new problems (*e.g.*, the design of a computer to determine which of two or more servos has control authority at any instant); systems of this type will not be considered here.

Another consideration in the selection of a gimbal configuration is the relative value of the three components of platform motion. For example, let w_x, w_y, and w_z be the x, y, and z components of the angular velocity of the platform with respect to the reference frame. If $w_x \gg w_y$, w_z, then it is advisable to align the outer gimbal axis as closely as possible with w_x. Then the inner gimbals are, to a large extent, isolated from the platform motion, reducing the velocity and acceleration requirements imposed on their servo motors. This concept is demonstrated with a numerical example following the discussion of Fig. 6.

Fig. 6 shows the six ways the three axes of a three-axis gimbal system can be arranged. It may be observed that

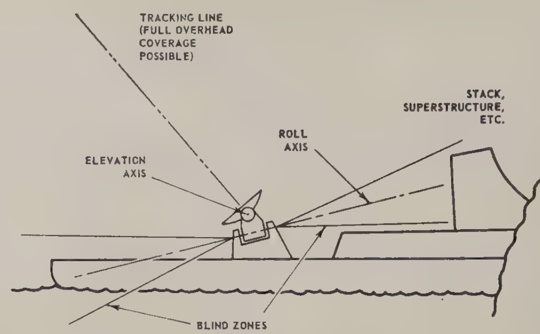


Fig. 10—Tilted-axis two-gimbal system.

a characteristic of these six configurations is that when all three gimbal angles are zero, the three axes are mutually orthogonal. It is possible to have a three-axis configuration, in which the inner and outer axes are parallel, when the middle gimbal angle is zero. Such a configuration may be termed a redundant system (*viz.*, the four-axis mount alluded to above) and is not considered here.

The servo rates required to stabilize each axis of the six gimbal configurations shown in Fig. 6 are listed in Table I. These rates are expressed in terms of w_x, w_y, w_z and the geometry which applies to each arrangement. It should be noted in each case by setting the inner gimbal angle equal to zero, the system reduces to a two-axis mount. For example, the three-axis mount in system II reduces to the two-axis mount discussed in Section VI, if $G23 \equiv 0$, and the elevation and train rates given in Table I correspond to (38) and (39), respectively. Because there are six configurations, most of which have no assigned notation, it is convenient to denote all the gimbal angles by the letter G, followed by two digits: the first runs from 1 through 6 and designates the system; the second runs from 1 through 3 and designates the location of the axis, with 1 being located between the platform and the outer gimbal, 2 between the outer and middle gimbals, and 3 between the middle gimbal and the inner gimbal, or controlled member.

The maximum servo rates associated with each axis of each type mount are also indicated in Table I, together with a recommended location of the gyros required for stabilization. The locations are determined so as to use single gimbal gyros for stabilization, and required AGC operators associated with these gyro locations, are also tabulated.

As an example of how Table I may be used, consider the case where

$$\overline{W}_{id}^{\,d} = \begin{vmatrix} 40° \text{ per second} \\ 4° \text{ per second} \\ 4° \text{ per second} \end{vmatrix}.$$

That is, the angular rate of the platform, in platform coordinates, is 40° per second about the x axis (roll) and 4° per second each about the y and z axes (pitch and yaw.)

[25] Cady, *et al.*, *op. cit.*, refer to more extensive studies of this problem performed by H. M. James, as for example, "Train Rates in Two-Axis Director," Rad. Lab., M.I.T., Cambridge, Mass., Rept. No. 8; September 18, 1943.

[26] R. S. Sanford, unpublished note.

TABLE I

SERVO RATES FOR THREE-AXIS MOUNTS

	Train			Roll			Elevation		
	Gyro Location	AGC Operator	Servo Rate / Maximum Servo Rate	Gyro Location	AGC Operator	Servo Rate / Maximum Servo Rate	Gyro Location	AGC Operator	Servo Rate / Maximum Servo Rate
System I	S_{G3}	1 sec $G12$ sec $G13$	$\frac{-w_z-\tan G12(w_x \sin G11-w_y \cos G11)}{-w_z-\tan G12(-w_x \cos -w_y \sin \alpha)}$	S_{G2}	2 1.0	$\frac{-(w_x \cos G11 + w_y \sin G11)}{-(w_x \cos_\alpha + w_y \sin \alpha)}$	S_{G3}	3 1.0	$\frac{\frac{1}{\cos G12}(w_x \sin G11-w_y \cos G11)}{\frac{1}{\cos G12}(w_x \cos \alpha-w_y \sin \alpha)}$
System II	S_{G3}	1 sec $G22$ sec $G23$	$\frac{-w_z-\tan G12(w_x \cos G21+w_y \sin G21)}{-w_z-\tan G12(w_x \cos \alpha+w_y \sin \alpha)}$	S_{G3}	3 1.0	$\frac{-\frac{1}{\cos G22}(w_x \cos G21+w_y \sin G21)}{-\frac{1}{\cos G22}(w_x \cos \alpha+w_y \sin \alpha)}$	S_{G3}	2 sec $G23$	$\frac{w_x \sin G21-w_y \cos G21}{-w_x \cos \alpha-w_y \sin \alpha}$
System III	S_{G3}	2 sec $G33$	$\frac{-(w_y \sin G31-w_z \cos G31)}{-(w_y \cos \gamma+w_z \sin \gamma)}$	S_{G1}	1 1.0	$\frac{-w_x-\tan G32(w_y \cos G31+w_z \sin G31)}{-w_x-\tan G32(w_y \cos \gamma+w_z \sin \gamma)}$	S_{G3}	3 1.0	$\frac{\frac{1}{\cos G32}(w_y \cos G31+w_z \sin G31)}{\frac{1}{\cos G32}(w_y \cos \gamma+w_z \sin \gamma)}$
System IV	S_{G3}	3 1.0	$\frac{\frac{1}{\cos G42}(w_y \sin G41-w_z \cos G41)}{\frac{1}{\cos G\,42}(-w_y \cos \gamma\ w_z \sin \gamma)}$	S_{G1}	1 1.0	$\frac{-w_x-\tan G42(w_y \sin G41-w_z \cos G41)}{-w_x-\tan G42(-w_y \cos \gamma-w_z \sin \gamma)}$	S_{G3}	2 sec $G43$	$\frac{-(w_y \cos G41+w_z \sin G41)}{-(w_y \cos \gamma+w_z \sin \gamma)}$
System V	S_{G3}	2 sec $G53$	$\frac{-(w_z \cos G51+w_x \sin G51)}{-(w_z \cos \delta+w_x \sin \delta)}$	S_{G3}	3 1.0	$\frac{\frac{1}{\cos G52}(w_z \sin G51-w_x \cos G51)}{\frac{1}{\cos G52}(-w_z \sin \delta-w_x \cos \delta)}$	S_{G3}	1 sec $G52$ sec $G53$	$\frac{-w_x-\tan G52(w_z \sin G51-w_x \cos G51)}{-w_y-\tan G52(w_z \sin \delta+w_x \cos \delta)}$
System VI	S_{G3}	3 1.0	$\frac{-\frac{1}{-\cos G62}(w_z \cos G61+w_x \sin G61)}{-\frac{1}{\cos G62}(w_z \cos \delta+w_x \sin \delta)}$	S_{G3}	2 1.0	$\frac{(w_z \sin G61-w_x \cos G61)}{-(w_z \cos \delta+w_x \sin \delta)}$	S_{G3}	1 sec $G62$ sec $G63$	$\frac{-w_y-\tan G62(w_z \sin G61+w_x \cos G61)}{-w_y-\tan G62(w_z \sin \delta+w_x \cos \delta)}$

Notes: 1) $w_x=(W_{id})^a_x$, $w_y=(W_{id})^a_y$, $w_z=(W_{id})^a_z$.
2) $\alpha=\tan^{-1} w_y/w_x$, $\gamma=\tan^{-1} w_z/w_y$. $\delta=\tan^{-1} w_x/w_z$.

Then, from Table I, the rates for each servo in all six cases may be determined as indicated in Table II. Note that the rates required for configurations III and IV are significantly smaller than those required for the other four types. This means that smaller servos and narrower servo bandwidths may be used in either the system III or IV, as compared with the other four configurations. Finally, it may be observed that if a conventional vertical gyro[27] is used in system IV (measuring pitch and roll) the stabilization orders from the director can be taken directly from the vertical gyro axes, without the use of a coordinate converter; this greatly simplifies the design of the system, and improves its accuracy and reliability. Consequently, the system IV gimbal configuration is recommended for this installation. The physical significance of this choice is clear from the sketch of system IV shown in Fig. 6. Since the outer gimbal rotates about the x axis of the vehicle, it "unwinds" the most violent component of the vehicle's motion, thus allowing the two remaining gimbals to move quite slowly. The letters, O, M, and I in the lower left-hand corner of each box designates the order of the gimbals from the base out. (See Fig. 6.) The quantities α, γ, and δ are defined on Table I. In all cases the middle gimbal angle is taken as 45°, considered representative of normal operation.

TABLE II
SERVO RATES, DEGREES PER SECOND

System (see Fig. 6)	Train	Roll	Elevation	Middle Gimbal Angle	Outer Gimbal Angle for Maximum Rate
I	45 *O*	41 *M*	58 *I*	$G12=45°$	$\alpha \approx 6°$
II	45 *O*	58 *I*	41 *M*	$G22=45°$	
III	6 *M*	46 *O*	8 *I*	$G32=45°$	$\gamma=45°$
IV	8 *I*	46 *O*	6 *M*	$G42=45°$	
V	41 *M*	58 *I*	45 *O*	$G52=45°$	$\delta=84°$
VI	58 *I*	41 *M*	45 *O*	$G62=45°$	

It may be observed that if the controlled member is to be isolated from the motion of the base, its angular acceleration must be zero as well as its angular velocity. That is, to geometrically stabilize the controlled member, it is required that

$$\overline{W}_{ia} = 0, \qquad D_i\overline{W}_{ia} = 0.$$

Knowing the angular acceleration for each servo, as well as its angular velocity, and given the dynamic accuracy requirements of the system, an estimate of the servo bandwidths may be determined. Further, from the gimbal inertias, required servo torques may be estimated. Then the designer can determine the required gear ratios, and make estimates of horsepower requirements. Two remarks should be made concerning these calculations. First, the angular acceleration has two components—one due to the relative angular accelerations, and the other, to products of angular rates (the Coriolis accelerations). Some of the problems involved in evaluating angular acceleration are discussed in Appendix II. Second, in determing the torque requirements care should be used in applying Newton's second law. For example, if the angular acceleration of the controlled member with respect to inertial space is zero, then the torque required to accelerate it is zero. This is true if a one-speed servomotor[28] is used; however, in most cases, the controlled member is driven through a gear train, and the acceleration of the motor armature with respect to inertial space may be considerable.

For example, assume that the controlled member is connected to the armature of a servo motor by a simple spur gear train. Then the axes of the motor, controlled member, etc., are all parallel, and from Newton's second law, the following scalar equations may be written:

$$J_c DW_{ic} = T_{gt} + T_d \tag{89}$$

$$J_M DW_{im} = T_m - NT_{gt} \tag{90}$$

$$J_p DW_{ip} = -T_m \tag{91}$$

where

J_c = moment of inertia of controlled member,
J_m = moment of inertia of servo motor,
J_p = moment of inertia of platform,
W_{ic} = angular velocity of controlled member with respect to Newtonian frame,
W_{im} = angular velocity of servo motor with respect to Newtonian frame,
W_{ip} = angular velocity of platform with respect to Newtonian frame,
T_{gt} = torque applied to controlled member by gear train,
T_d = disturbing torque applied to controlled member,
T_m = torque applied to gear train by motor,
N = gear ratio,
$D(\ ) = dt/dt\ (\ )$.

Further, the kinematic relations are

$$W_{im} = W_{ip} + W_{pm} \tag{92}$$

$$W_{ic} = W_{ip} + W_{pc} \tag{93}$$

$$W_{pc} = NW_{pm}. \tag{94}$$

Eq. (92) states that the angular rate of the motor with respect to the Newtonian frame is equal to the angular rate of the platform with respect to the Newtonian

[27] L. Becker, "Gyro pickoff indications at arbitrary plane attitudes," *J. Aeronaut. Science*, vol. 18, pp. 718–724; November, 1951.
M. J. Abzug, "Application of matrix operators to the kinematics of airplane motion," *J. Aeronaut. Sciences*, vol. 23, pp. 679–684; July, 1956.

[28] F. M. Bailey, "Performance of drive members in feedback control systens," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-1, p. 74; May, 1956.

frame, plus the angular rate of the motor with respect to the platform. Eq. (93) states that the angular rate of the controlled member with respect to the Newtonian frame is equal to the angular rate of the platform with respect to the Newtonian frame, plus the angular rate of the controlled member with respect to the platform. Eq. (94) states that the angular rate of the controlled member with respect to the platform is N times the angular rate of the motor with respect to the platform; N is the gear ratio, a number much smaller than one as defined here.

The servomotor functions so as to make W_{ic} and DW_{ic} approach zero; that is

$$W_{ic} \to 0 \qquad DW_{ic} \to 0$$

so that

$$W_{ip} \to -W_{pc} \qquad T_{gt} \to -T_d. \tag{95}$$

Therefore, combining (90), (92), (94), and (95) gives

$$T_m = -NT_d - \frac{1-N}{N} J_m DW_{cp}. \tag{96}$$

For $N \ll 1$, (96)

$$T_m \approx \frac{J_m DW_{ip}}{N} \tag{97}$$

and it may be observed that while $J_m DW_{ip}$ may not be absolutely large, the factor $1/N$ may be considerable, so that the required motor torque T_m is usually not negligible.

Conclusion of Part 2

The vector algebra described thus far has been shown suitable for the manipulation of three-dimensional position vectors (Part 1) which may be used to design coordinate converters, and for the treatment of three-dimensional angular velocities (Part 2), which may be used to design stabilization systems. In the third and concluding part of this paper, we will extend our vector algebra to problems in kinetics; that is, we will examine the forces and moments which produce the motions analyzed in Parts 1 and 2. Such studies are of interest in the design of stabilization systems, but the primary emphasis in Part 3 is on gyroscopic phenomena. Gyroscopic devices are of special interest since they play such a basic role in all of the devices described so far, and, in Part 3, the equations which described the performance of these devices will be developed in detail.

Appendix I

Instrumenting the Coordinate Converter

In Part 1 the principles which govern the design of coordinate converters were examined. It was shown that the design of such devices can be accomplished by the use of plane geometry and implicit computation techniques without the necessity of explicitly solving the pertinent equations of spherical trigonometry. The purpose of Appendix I is to describe the electrical components which may be used in instrumenting such computers, and to indicate some of the detailed problems which are involved in the design of these devices. In addition, a proof of the implicit computation technique presented in Part 1 is given.

Figs. 11 and 12 show two components which may be used to instrument coordinate converters.[29] Fig. 11 represents an electromagnetic resolver, capable of resolving a plane vector from one set of Cartesian coordinates to another. Let θ represent the position of the shaft, analogous to the rotation of Cartesian coordinates about a common axis; further, let the vector in one set of coordinates be represented by the ac voltages Ex_1 and Ey_1. Ex_1 and Ey_1 are the analogs of the x and y components of the vector in S_1. The electromagnetic resolver produces the two output voltages Ex_2 and Ey_2 such that

$$Ex_2 = Ex_1 \cos\theta + Ey_1 \sin\theta$$
$$Ey_2 = -Ex_1 \sin\theta + Ey_1 \cos\theta$$

so that Ex_2 and Ey_2 are the ac voltage analogs of the x and y components of the vector in S_2.

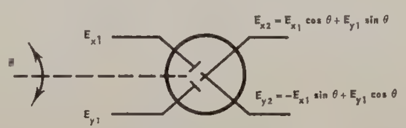


Fig. 11—Electromagnetic resolver.

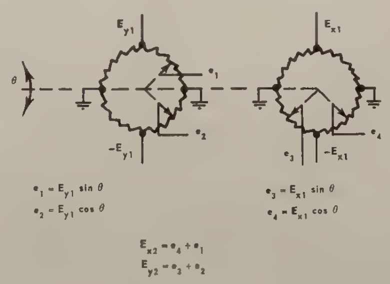


Fig. 12—Potentiometer resolver.

Usually, when a number of such devices are cascaded together to form a coordinate converter, buffer amplifiers are inserted between stages to prevent adverse loading and coupling effects. A number of manufacturers, however, have developed components which do not require such isolation; in fact, most of these companies provide complete resolver chains built to the customers' requirements. These resolver chains include carefully designed mechanical structures and gear trains integrated into highly compact packages, operating with great precision over a wide temperature range.

[29] Cady, *et al.*, *op. cit.*, pp. 123–126, and Greenwood, *et al.*, *op. cit.*; the latter reference is an excellent source work for problems of this nature and includes a good discussion on implicit computation techniques as well.

The device shown schematically in Fig. 12 makes use of two nonlinear potentiometers, whose resistances are so arranged that the output voltages vary sinusoidally with the rotation of the shaft. In this device, it should be noted that voltage-summing devices are required to properly form the two components of the output vector. Note, also, that the sine taps of the two potentiometers must be mechanically phase 180° apart.

Potentiometers can be used with either ac or dc voltages; consequently they are frequently used in dc analog computers. One of the difficulties encountered in the use of sine-cosine potentiometers is the resolution of voltage from wire to wire and the action of the wiper, both of which result in noisy operation. In computer use, large diameter mandrils minimize the effects of resolution errors, and, in addition, the use of deposited materials in place of wire winding greatly minimizes the problem of noise.

Fig. 13 shows a partial space-flow diagram for a coordinate converter, and Fig. 14 illustrates in block diagram form the servo driven resolver which performs the operation indicated by Fig. 13. The vector equation representing the ideal operation of the converter is given by

$$\overline{U}^a = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

$$= T_{ab} \begin{vmatrix} x_b \\ 0 \\ z_b \end{vmatrix}$$

$$= T_{ab}T_{bd} \begin{vmatrix} x_d \\ y_d \\ z_d \end{vmatrix} \text{ etc.}$$

To analyze the servo performance, let $Ed' = (Ed')_0 + \Delta Ed'$, where $(Ed')_0$ is some equilibrium value, and $\Delta Ed'$ is a perturbation about that value, to which the usual trigonometric approximations can be applied, namely,

$$\sin \Delta Ed' \approx \Delta Ed',$$

$$\cos \Delta Ed' \approx 1.$$

Then it follows that

$$(x_a)_0 = 1 = (x_b)_0 \cos (Ed')_0 - (z_b)_0 - \sin (Ed')_0$$

$$(z_a)_0 = 0 = (x_b)_0 \sin (Ed')_0 + (z_b)_0 \cos (Ed')_0$$

so that

$$(x_b)_0 = \cos (Ed')_0$$

$$(z_b)_0 = -\sin (Ed')_0.$$

From Fig. 14, the servo relations can be written as

$$(Ed')_0 + \Delta Ed' = Y(D) z_a \tag{98}$$

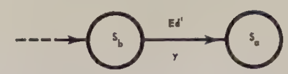


Fig. 13—Partial space-flow diagram for coordinate converter.

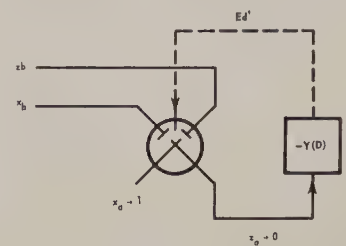


Fig. 14—Partial servo block diagram for coordinate converter.

where

$$z_a = \cos (Ed')_0 \sin [(Ed')_0 + \Delta Ed'] - \sin (Ed')_0 \cos [(Ed')_0 + \Delta Ed']$$

$$Y(D) = \text{servo transfer function.}$$

For $\Delta Ed'$ small, (98) becomes

$$\Delta Ed' \approx \frac{(Ed')_0}{Y(D) + 1}. \tag{99}$$

A typical form for $Y(D)$ is given by

$$Y(D) = \frac{K_v}{D(TD + 1)}$$

so that (99) may be written as

$$\Delta Ed' = \frac{D(TD + 1)(Ed')_0/K_v}{\frac{T}{K_v} D^2 + \frac{1}{K_v} D + 1}. \tag{100}$$

If $(Ed')_0$ changes at a constant rate W, the equilibrium value of $\Delta Ed'$ is given by

$$(\Delta Ed')_{fv} = -\frac{W}{K_v}.$$

Therefore, as long as W/K_v is sufficiently small, so that the small angle approximations are valid, the resolver loop is stable and the usual linear design techniques for the selection of K_v and T are applicable.

When the small angle approximation is not used, it can be seen that the system behaves like a pendulum, with a sinusoidally varying restoring force. Ku,[30] among others, considers this problem in detail, and the interested reader is referred to this work for a more extensive analysis.

In Section III of Part 1 a coordinate converter was considered which was characterized by the inputs E and B, the disturbances Eio and Zo and the outputs Bd' and Ed'. Eq. (11) stated that to solve for Bd' and Ed' in terms of E, B, Eio, and Zo, it is necessary to solve the following matrix equation:

[30] Y. H. Ku, "Analysis and Control of Non-Linear Systems," The Ronald Press Co., New York, N. Y., pp. 69 ff; 1958.

$$T_{at} = \begin{vmatrix} 1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_2 \end{vmatrix}. \tag{101}$$

Since $T_{at}T_{ta} = I$, $a_2 = a_3 = 0$, so that (101) becomes

$$T_{at} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix} \tag{102}$$

and T_{at} is given by

$$T_{at} = T_{ab}T_{bd}T_{dr}T_{ri}T_{ih}T_{ht}$$

where

$$T_{ab} = \begin{vmatrix} \cos Ed' & 0 & -\sin Ed' \\ 0 & 1 & 0 \\ \sin Ed' & 0 & \cos Ed' \end{vmatrix}$$

$$T_{bd} = \begin{vmatrix} \cos Bd' & \sin Bd' & 0 \\ -\sin Bd' & \cos Bd' & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$T_{dr} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zo & \sin Zo \\ 0 & -\sin Zo & \cos Zo \end{vmatrix}$$

$$T_{ri} = \begin{vmatrix} \cos Eio & 0 & -\sin Eio \\ 0 & 1 & 0 \\ \sin Eio & 0 & \cos Eio \end{vmatrix}$$

$$T_{ih} = \begin{vmatrix} \cos B & -\sin B & 0 \\ \sin B & \cos B & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$T_{ht} = \begin{vmatrix} \cos E & 0 & \sin E \\ 0 & 1 & 0 \\ -\sin E & 0 & \cos E \end{vmatrix}.$$

The solution of (102) is simplified by noting that

$$T_{at} = T_{ab}T_{bt} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix}$$

or

$$T_{bt} = T_{ba}\begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{vmatrix}$$

$$T_{bt} = \begin{vmatrix} \cos Ed' & c_2 \sin Ed' & c_3 \sin Ed' \\ 0 & b_2 & b_3 \\ -\sin Ed' & c_2 \cos Ed' & c_3 \cos Ed' \end{vmatrix}. \tag{103}$$

Now, let

$$T_{dt} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

so that

$$T_{bt} = T_{bd}T_{dt}$$

$$= \begin{vmatrix} A_1 \cos Bd' + B_1 \sin Bd' & x & x \\ -A_1 \sin Bd' + B_1 \cos Bd' & x & x \\ C_1 & x & x \end{vmatrix} \tag{104}$$

where the x's represent quantities which are of no interest here, since, by equating (103) and (104),

$$Ed' = \sin^{-1}(-C_1) \tag{105}$$

$$Bd' = \tan^{-1}\left(\frac{B_1}{A_1}\right). \tag{106}$$

That is, in order to determine the output quantities Ed' and Bd' in terms of the inputs E and B, and the disturbances Eio and Zo, we need only to insert the values of A_1, B_1 and C_1 into (105) and (106) as indicated.

Consequently, the evaluation of T_{dt} is considerably simplified because we seek only to find the first column of the matrix, made up of A_1, B_1, and C_1.

The transformation matrix T_{dt} may be evaluated by noting that

$$T_{dt} = T_{dr}T_{rt}$$

or

$$T_{rt} = T_{rd}T_{dt}$$

$$= \begin{vmatrix} A_1 & x & x \\ B_1 \cos Z_0 - C_1 \sin Z_0 & x & x \\ B_1 \sin Z_0 + C_1 \cos Z_0 & x & x \end{vmatrix}.$$

That is, it is sufficient, for our purpose, to evaluate only the first column of T_{rt}, as follows:

$$T_{it} = T_{ih}T_{ht} = \begin{vmatrix} \cos E \cos B & -\sin B & \cos B \sin E \\ \cos E \sin B & \cos B & \sin B \sin E \\ -\sin E & 0 & \cos E \end{vmatrix}$$

$$T_{rt} = T_{ri}T_{it}$$

$$= \begin{vmatrix} \cos Eio \cos B \cos E + \sin Eio \sin E & x & x \\ \cos E \sin B & x & x \\ \sin Eio \cos E \cos B - \sin E \cos Eio & x & x \end{vmatrix}$$

so that

$$A_1 = \cos Eio \cos B \cos E + \sin Eio \sin E,$$

$$B_1 = \cos Zo \cos E \sin B$$
$$+ \sin Zo(\sin Eio \cos E \cos B - \sin E \cos Eio),$$

$$C_1 = -\sin Zo \cos E \sin B$$
$$+ \cos Zo(\sin Eio \cos E \cos B - \sin E \cos Eio).$$

Therefore, from (105)

$$Ed' = \sin^{-1}[\sin Zo \cos E \sin B + \cos Zo(\sin E \cos Eio - \sin Eio \cos E \cos B)] \quad (107)$$

and from (106),

$$Bd' = \tan^{-1} \frac{\cos Zo \cos E \sin B + \sin Zo\,(\sin Eio \cos E \cos B - \sin E \cos Eio)}{\cos Eio \cos B \cos E + \sin Eio \sin E}. \quad (108)$$

Comparison of (107) with (21) in Part 1, and (108) with (17) in Part 1 shows that the technique of implicit solution given in Section III is valid.

Appendix II

Eulerian Angles and Vector Differentiation

In Part 1, it was demonstrated that a vector could be transformed, via a series of planar rotations, from one set of three-dimensional coordinates to another. Goldstein[31] points out that three such rotations serve as three independent parameters which can be used to specify the attitude of a rigid body in space with respect to some fixed reference frame. Classically, three such angles, suitably chosen, are known as "Eulerian angles." However, as Goldstein points out,[32] there is no general agreement as to how the Eulerian angles are defined. Consequently, for the convenience of this paper, we have generalized this concept and designated as "Euler angles" any three successive rotations which uniquely and independently define the attitude of a rigid body in space.

In particular, we will consider here three angles which are of special interest in naval and aircraft systems. These three angles uniquely determine the attitude of ship, airplane, etc., with respect to some arbitrary reference, and, in addition, two of these angles are directly measurable with a standard vertical gyro.[27,33] These angles are represented by the symbols Co, Eio, and Zo, where

$$Co = \text{Heading angle},^{19,34}$$

$$Eio = \text{Pitch angle},$$

$$Zo = \text{Roll angle}.$$

Fig. 15 shows a space-flow diagram utilizing those three Euler angles to define the attitude of a body in space.

The set $(xyz)_i$ represented by S_i on Fig. 15 defines the reference space, while the set $(xyz)_d$, represented by S_d on Fig. 15, designates the rotating body. S_p and S_r represent auxiliary frames.

Fig. 15—Space-flow diagram for Euler angles.

It is of interest to express the angular rate of the body in terms of the rates of change of the three Euler angles. To do this, we note that

$$\overline{W}^{d}_{id} = T_{dr}T_{rp}\overline{W}^{p}_{ip} + Td_r\overline{W}^{r}_{pr} + \overline{W}^{d}_{rd}$$

where

$$\overline{W}^{d}_{rd} = \begin{vmatrix} DZo \\ 0 \\ 0 \end{vmatrix} \qquad \overline{W}^{r}_{pr} = \begin{vmatrix} 0 \\ DEio \\ 0 \end{vmatrix}$$

$$\overline{W}^{p}_{ip} = \begin{vmatrix} 0 \\ 0 \\ DCo \end{vmatrix}$$

$$T_{rp} = \begin{vmatrix} \cos Eio & 0 & -\sin Eio \\ 0 & 1 & 0 \\ \sin Eio & 0 & \cos Eio \end{vmatrix}$$

$$T_{dr} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zo & \sin Zo \\ 0 & -\sin Zo & \cos Zo \end{vmatrix}.$$

Performing the indicated operations and combining results in the following three scalar equations, representing the body axis components of the angular rate of the body with respect to S_i as expressed in terms of the Euler angles Co, Eio, and Zo,

$$W_x = DZo - DCo \sin Eio, \quad (109)$$

$$W_y = DCo \cos Eio \sin Zo + DEio \cos Zo, \quad (110)$$

$$W_z = DCo \cos Eio \cos Zo + DEio \sin Zo. \quad (111)$$

Given the angular rate of a platform with respect to inertial space, as might be measured by three rate gyros for example, it is of interest to express the time rates of change of the three Euler angles Co, Eio, and Zo in terms of W_x, W_y, and W_z. This may be done as follows: add the product of sin Zo (110) to the product of cos Zo

[31] Goldstein, *op. cit.*, pp. 107–109.
[32] *Ibid.*, p. 108.
[33] See Part 3, to be published.
[34] These definitions are taken from OP-1700, except that Co, in OP-1700, designates direction from true north. This restriction is not required here and is not implied. Furthermore, as previously stated, the sign of these angles is given by the right-hand convection.

(111) to yield (112); subtract the product of cos Zo (110) from the product of sin Zo (111) to yield (113); combine (109) and (112) to produce (114):

$$DCo = \frac{W_z \cos Zo + W_y \sin Zo}{\cos Eio} \tag{112}$$

$$DEio = W_y \cos Zo - W_y \sin Zo, \tag{113}$$

$$DZo = W_x + (W_z \cos Zo + W_y \sin Zo) \tan Eio. \tag{114}$$

When dealing with coordinate systems which rotate with respect to each other, as in the cases considered in this paper, it is most important to exercise extreme care in performing differentiation and integration operations on vectors. For example, let $\overline{V}(t)$ be a vector quantity which varies with time, and $D=d/dt(\ )$. Note that the operation $D[\overline{V}(t)]$ is ambiguous, since the time rate of change of $\overline{V}(t)$ varies in accordance with the coordinate frame to which this motion is referred; to make the differentiation operation meaningful, it is necessary to indicate, by means of a subscript to the operator D, for example, from what set of coordinates $D[\overline{V}(E)]$ is to be measured. That is

$$D[\overline{V}(t)] = ?$$

whereas

$$D_a[\overline{V}(t)] =$$

$$D_a\left[\,|i_a j_a k_a| \begin{vmatrix} V_x \\ V_y \\ V_z \end{vmatrix}\,\right] = |i_a j_a k_a| \begin{vmatrix} DV_x \\ DV_y \\ DV_z \end{vmatrix}$$

$$= \text{time rate of change of } \overline{V}(t) \text{ with respect to } S_a,$$

where i_a, j_a, and k_a are the unit vectors of S_a, and V_x, V_y, and V_z are the x, y, and z components of $\overline{V}(t)$ in S_a, respectively.

Frequently, it is necessary or convenient to determine the time rate of change[35] of a vector with respect to, say, S_a, in terms of its time rate of change with respect to S_b. The rule which expresses these relations is sometimes referred to as the Law of Coriolis, and may be written as[36]

$$D_a\overline{V} = D_b\overline{V} + \overline{W}_{ab} \times \overline{V} \tag{115}$$

where

$$\overline{V} = \text{any vector},$$

$$D_a\overline{V} = \frac{d}{dt}(\overline{V})_a,$$

$$D_b\overline{V} = \frac{d}{dt}(\overline{V})_b,$$

$$\overline{W}_{ab} = \text{angular rate of } S_b \text{ with respect to } S_a.$$

[35] These statements are equally true for differentiation with respect to any other independent variable.
[36] Goldstein, *op. cit.*, pp. 132–140.

Fig. 16 illustrates the geometric significance of (115). Let S_a be the reference space. The rotation of S_b with respect to S_a, denoted by $\overline{W}_{ab}$, is indicated in Fig. 16 by the vertical arrow. If $\overline{V}$ is a vector in S_b, then $D_b\overline{V}$ is a vector *along* $\overline{V}$ and rotates with respect to S_a. Since S_b is rotating with respect to S_a, $\overline{V}$ changes direction with time when viewed from S_a. This change is normal to $\overline{V}$, and is given by $\overline{W}_{ab} \times \overline{V}$. If $\overline{V}$ is a position vector, then the magnitude of the velocity of $\overline{V}$ due to $\overline{W}_{ab}$ is given by $|\overline{W}_{ab}|\ |\overline{V}| \sin \alpha$, where α is the angle between $\overline{W}_{ab}$ and $(\overline{V})$. The direction of $\overline{W}_{ab} \times \overline{V}$ is normal to both $\overline{W}_{ab}$ and to $\overline{V}$. Therefore, $D_a\overline{V}$ consists of two components: one along $\overline{V}$, proportional to the rate at which $\overline{V}$ is changing in length, and the other normal to $\overline{V}$, proportional to the rate with which $\overline{V}$ is rotating with respect to S_a.

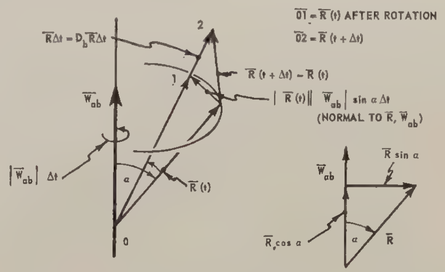


Fig. 16—Differentiation of rotating vector.

Frequently, vectors exist which are products of a vector and a scalar. For example, let

$$\overline{V}(t) = S(t)\overline{U}(t)$$

so that

$$D_a\overline{V}(t) = D_a[S(t)\overline{U}(t)]$$

$$= D[S(t)]\overline{U}(t) + S(t)D_a[\overline{U}(t)]. \tag{116}$$

That is, $D_a[S(t)] = D[S(t)]$, since the rate of change of a scalar is independent of the reference frame from which this change is measured.

As an example of operations of this kind, consider the angular acceleration of a body in terms of its Euler angles. Referring to Fig. 15, this angular acceleration may be written as follows:

$$\overline{A}^d_{id} = D_i\overline{W}^d_{id}$$

where the components of $\overline{W}_{idd}$ are given by (109)–(111) above. Expanding A_{idd} we have

$$\overline{A}^d_{id} = D_i(T_{dr}T_{rp}\overline{W}^p_{ip} + T_{dr}\overline{W}^r_{pr} + \overline{W}^d_{rd}). \tag{117}$$

Each of the terms of (117) may be expanded by the application of (115) and (116) as follows:

$$D_i(T_{dr}T_{rp}\overline{W}_{ip}^{p}) = D(T_{dr})T_{rp}\overline{W}_{ip}^{p}$$

$$+ DZo\begin{vmatrix} 0 & 0 & 0 \\ 0 & -\sin Zo & \cos Zo \\ 0 & -\cos Zo & -\sin Zo \end{vmatrix}\begin{vmatrix} \cos Eio & 0 & -\sin Eio \\ 0 & 1 & 0 \\ \sin Eio & 0 & \cos Eio \end{vmatrix}\begin{vmatrix} 0 \\ 0 \\ DCo \end{vmatrix}$$

$$+ DEio\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zo & \sin Zo \\ 0 & -\sin Zo & \cos Zo \end{vmatrix}\begin{vmatrix} -\sin Eio & 0 & -\cos Eio \\ 0 & 0 & 0 \\ \cos Eio & 0 & -\sin Eio \end{vmatrix}\begin{vmatrix} 0 \\ 0 \\ DCo \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zo & \sin Zo \\ 0 & -\sin Zo & \cos Zo \end{vmatrix}\begin{vmatrix} \cos Eio & 0 & -\sin Eio \\ 0 & 1 & 0 \\ \sin Eio & 0 & \cos Eio \end{vmatrix}\begin{vmatrix} 0 \\ 0 \\ D^2Co \end{vmatrix}$$

$$= \begin{vmatrix} -D^2Co \sin Eio \quad - DCoDEio \cos Eio \\ D^2Co \cos Eio \sin Zo + DZoDCo \cos Eio \cos Zo - DEioDCo \sin Eio \sin Zo \\ D^2Co \cos Eio \cos Zo - DZoDCo \cos Eio \sin Zo - DEioDCo \sin Eio \cos Zo \end{vmatrix}. \tag{118}$$

$$D_i(T_{dr}\overline{W}_{pr}^{r}) = D(T_{dr})\overline{W}_{pr}^{r} + T_{dr}[D_r\overline{W}_{pr}^{r} + \overline{W}_{ir}^{r} \times \overline{W}_{pr}^{r}]$$

$$= DZo\begin{vmatrix} 0 & 0 & 0 \\ 0 & -\sin Zo & \cos Zo \\ 0 & -\cos Zo & -\sin Zo \end{vmatrix}\begin{vmatrix} 0 \\ DEio \\ 0 \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Zo & \sin Zo \\ 0 & -\sin Zo & \cos Zo \end{vmatrix}\left\{\begin{vmatrix} 0 \\ D^2Eio \\ 0 \end{vmatrix} + \begin{vmatrix} -DCo \sin Eio \\ DEio \\ DCo \cos Eio \end{vmatrix} \times \begin{vmatrix} 0 \\ DEio \\ 0 \end{vmatrix}\right\}$$

$$= \begin{vmatrix} -DEioDCo \cos Eio \\ D^2Eio \cos Zo \quad - DEioDZo \sin Z_0 - DEioDCo \sin Eio \sin Zo \\ -D^2Eio \sin Zo \quad - DEioDZo \cos Z_0 - DEioDCo \sin Eio \cos Zo \end{vmatrix}. \tag{119}$$

$$D_i\overline{W}_{rd}^{d} = D_d\overline{W}_{rd}^{d} + \overline{W}_{id}^{d} \times \overline{W}_{rd}^{d} = \begin{vmatrix} D^2Zo \\ 0 \\ 0 \end{vmatrix} + \begin{vmatrix} W_x \\ W_y \\ W_z \end{vmatrix} \times \begin{vmatrix} DZo \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} D^2Zo \\ +DZoW_z \\ -DZoW_y \end{vmatrix}. \tag{120}$$

Combining Equations (110), (111), and (118)–(120) gives

$$A_{id}^{d} = \begin{vmatrix} A_1 \\ A_2 \\ A_3 \end{vmatrix}$$

with

$$A_1 = (D^2Zo - D^2Co \sin Eio) - (2DCoDEio \cos Eio), \tag{121a}$$

$$A_2 = (D^2Co \cos Eio \sin Zo + D^2Eio \cos Zo) + (2DCoDZo \cos Eio \cos Zo - 2DCoDEio \sin Eio \sin Zo - 2DEioDZo \sin Zo), \tag{122a}$$

$$A_3 = (D^2Co \cos Eio \cos Zo - D^2Eio \sin Zo) - (2DCoDEio \sin Eio \cos Zo + 2DCoDZo \cos Eio \sin Zo + 2DEioDZo \cos Zo). \tag{123a}$$

The following observations may be made concerning (121a), (122a) and (123a).

1) In each case, the first term on the right side of the equation is formally similar to (109)–(111), respectively.
2) In each case, the second term on the right hand side of the equation has the familiar form of the "Coriolis" acceleration—that is, it is proportional to twice the product of two angular velocities.

Consequently, we write (121a)–(123a) as follows:

$$A_1 = A_x + (\mathrm{ACA})_x \tag{121b}$$

$$A_2 = A_y + (\mathrm{ACA})_y \tag{122b}$$

$$A_3 = A_z + (\mathrm{ACA})_z \tag{123b}$$

where

$$A_x = D^2Zo - D^2Co \sin Eio$$

$$A_y = D^2Co \cos Eio \sin Zo + D^2Eio \cos Zo$$

$$A_z = D^2Co \cos Eio \cos Zo - D^2Eio \sin Ao$$

and we read ACA as Angular Coriolis Acceleration, with the x, y, and z subscripts denoting the x, y, and z components, respectively so that

$$(\mathrm{ACA})_x = -2(DCoDEio \cos Eio)$$

$$(\mathrm{ACA})_y = 2(-DCoDEio \sin Eio \sin Zo + DCoDZo \cos Eio \cos Zo - DEioDZo \sin Zo)$$

$$(\mathrm{ACA})_z = 2(-DCoDEio \sin Eio \cos Zo - DCoDZo \cos Eio \sin Zo - DEioDZo \cos Zo).$$

Root Locus Properties and Sensitivity Relations in Control Systems*

HANOCH UR†

Summary—**The differential properties of root loci including pole sensitivity, angle of slope, and curvature at ordinary and irregular points are investigated in a unified manner. A relation between the sensitivity function and pole sensitivity is established. The sensitivity is shown to determine variations in the transfer function due to large (not only infinitesimal) variations in K. Additional properties of loci which are developed include loci of a variable pole position and the existence of asymptotes for open-loop transfer functions with no poles or zeros at infinity.**

The locus is treated as a transformation of a line (the real axis) in the K plane to the s plane, and properties of analytic functions are used to simplify calculations and results. It is shown that the properties obtained can be extended to the general root locus of a nonreal K.

Introduction

In the study of linear systems it is interesting to investigate the effects of variations of certain elements, or parameters, on the network function. It has been noted by Bode[1] that one can write the network function in a bilinear form,

$$T(s) = \frac{A(s) + KB(s)}{C(s) + KD(s)}, \tag{1}$$

where K is the variable element. A, B, C, and D are functions of all other fixed elements or parameters, and the complex variable s. In problems involving control systems, a commonly encountered special case occurs when $A=0$ and $B=D$.

Although the transmittance $T(s)$ is written formally as a function of s alone, it should be noted that it is actually a function of s and K. Bode investigated the differential changes in T due to small changes in K and introduced the term "sensitivity" commonly defined now as the logarithmic derivative of T with respect to the logarithm of K.

$$S_K^T \triangleq \frac{d \ln T}{d \ln K} = \frac{\frac{dT}{T}}{\frac{dK}{K}}. \tag{2}$$

In recent years, with the increased utilization of the pole-zero approach, it has become increasingly important to examine variations in position of the poles and zeros of the network function due to changes in network parameters.

This work will consider the effect that variation of K will have on the poles and zeros of T, both in the "large" and in the "small." The effects in the large, or the properties of root loci, are better known and more investigated than the effects in the small, *i.e.*, differential properties which will be discussed here in greater detail.

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[1] H. W. Bode, "Network Analysis and Feedback Amplifiers Design," D. Van Nostrand Co., Inc., New York, N. Y., p. 223; 1945.

Both the numerator and denominator of T are of the form

$$H(s, K) = q(s) + Kp(s) \tag{3}$$

where K is the variable parameter and $p(s)$ and $q(s)$ are polynomials. The roots of $H(s, K)=0$, which characterize H, therefore, depend upon the value of K. $H(s, K)=0$ constitutes an implicit relationship between s and K, and as K takes on all positive values, a number of curves in the s plane are obtained. These curves are the loci of the roots of H, *i.e.*, the "root loci."

The root locus was introduced by Evans[2] in 1948. He used it mainly for the investigation of control systems with a transfer function T, of the form

$$T = \frac{G}{1+G} = \frac{Kp(s)}{q(s) + Kp(s)} \tag{4}$$

where $G = Kp/q$ is the forward gain in a unity feedback control system. For such systems, only the denominator is of interest. As a result, there arose the generally accepted, but unfortunately confusing terminology, in which the zeros of p and q were called open-loop "poles" and "zeros," respectively, and those of H, "closed-loop poles." Root locus techniques are useful both as an analytical tool and as an aid to the rapid evaluation of control system characteristics. Rules helpful in drawing of root loci quickly were established.[3]

An alternate approach is obtained by viewing the relationship between s and K as a conformal mapping[4] or transformation. The root locus then becomes a mapping of the real axis of the K plane into the s plane (the definition can be extended to other lines). The characterization of T in terms of its critical points (poles and zeros) has led to the investigation of the differential relationship between K and the s_j[5–9] where the s_j are the solutions of the equation $H=0$.

The relationship, known as the pole-zero sensitivity, takes the place of the previously defined sensitivity.

[2] W. R. Evans, "Graphical analysis of control systems" *Trans. AIEE*, vol. 67, pt. 1, pp. 547–551; 1948.

[3] J. G. Truxal, "Automatic Feedback Systems Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y., ch. 4; 1955.

[4] F. M. Reza, "Some mathematical properties of root loci for control systems design," *Trans. AIEE, Commun. and Electronics*, vol. 75, pt. 1, pp. 103–108; March, 1956.

[5] The roots of $H=0$ are denoted as s_j to distinguish them from the variable s which may, of course, assume any value. It also emphasizes the fact that $H=0$ may be satisfied by several values of s.

[6] A. Papoulis, "Displacement of the zeros of the impedance $Z(p)$ due to incremental variations in the network elements," Proc. IRE, vol. 43, pp. 79–82; January, 1955.

[7] J. G. Truxal and I. Horowitz, "Sensitivity Considerations in Active Synthesis," presented at Midwest Conf. on Circuit Theory; 1956.

[8] E. J. Angelo, Jr., "Design of Feedback Systems," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Rept. No. R-449-55; 1955.

[9] H. Ur, "Geometrical Properties of Root Loci," M.S. thesis, Purdue University, Lafayette, Ind.; August, 1956.

The Relationship Between Sensitivity and Pole Sensitivity

In order to establish the relationship between sensitivity and pole sensitivity, let

$$T(s) = \frac{g \prod_{1}^{n} (s + z_i)}{\prod_{1}^{n} (s + p_i)} \tag{5}$$

and let the z_i, p_i and g be functions of some parameter x (temperature, gain of some element, etc.).

The logarithmic derivative of T with respect to x is

$$\frac{dT}{T} = dx\left[\frac{1}{g}\frac{\partial g}{\partial x} + \sum_{1}^{m} \frac{\frac{\partial z}{\partial x}}{s + z_i} - \sum_{1}^{n} \frac{\frac{\partial p_i}{\partial x}}{s + p_i}\right]. \tag{6}$$

On dividing both sides by dx/x

$$S_x^T = \frac{\frac{dT}{dx}}{\frac{T}{x}} = \frac{\frac{\partial g}{\partial x}}{\frac{g}{x}} + \sum_{1}^{m} x \frac{\partial z_i}{\partial x} \frac{1}{s + z_i} - \sum_{1}^{n} x \frac{\partial p_i}{\partial_x} n^x \frac{1}{s + p_i}. \tag{7}$$

Eq. (7) gives the relationship between the classical definition of sensitivity and the pole and zero sensitivities. It confirms the intuitive feeling that a small value of S_x^T would be achieved, for a given value of s, if the p_i and z_i in the neighborhood of s were associated with a low pole-zero sensitivity. The pole-zero sensitivities can be defined in several ways, but in light of (7),

$$S_x^{p_i} \triangleq x \frac{\partial p_i}{\partial x}$$

$$S_x^{z_i} \triangleq x \frac{\partial z_i}{\partial x} \tag{8}$$

is one apparently useful choice.

This definition is not necessarily the most useful for all situations. Occasionally, an alternate definition may be necessary, meaningful or useful (see Appendix II).

The Pole Sensitivity

In the analysis of the root locus, the question of its behavior "in the small"—around a particular point s_j—is of interest. This investigation starts with the examination of ordinary points where the locus has a definite derivative and mapping between s_j and K is conformal.

The root locus is defined by

$$H(s, K) \triangleq q(s) + Kp(s) = 0; \tag{9}$$

hence

$$K = -\frac{q(s)}{p(s)} = -\prod_i (s_j - s_i)^{A_i} \tag{10}$$

where $A_i = 1$ if s_i is a root of q and $A_i = -1$ if s_i is a root of p.

The logarithmic derivative of (10) is

$$\frac{\frac{dK}{K}}{ds_j} = +\frac{q'}{q} - \frac{p'}{p} = +\sum_i \frac{A_i}{s_j - s_i}. \tag{11}$$

The reciprocal of (11), the pole sensitivities with respect to K, is

$$S_K^p = \frac{ds_j}{\frac{dK}{K}} = \frac{1}{\sum_i \frac{A_i}{s_j - s_i}} = \frac{-pq}{qp' - pq'}$$

$$= -\frac{Kp}{q' + Kp'}. \tag{12}$$

it follows from (10) and (12)

$$S_K^p = -\frac{Kp}{q' + Kp'} = -\frac{Kp}{\frac{\partial H}{\partial s}} = -k_j \tag{13}$$

where k_j is the residue expressed in (4) at the point s_j.[6,8,10] The same result can be obtained directly from (9) by

$$\frac{ds_j}{\frac{dK}{K}} = -K\frac{\frac{\partial H}{\partial K}}{\frac{\partial H}{\partial s_j}} = -\frac{Kp}{\frac{dq}{ds} + K\frac{dp}{ds}}. \tag{14}$$

Forms (12) of S_K^p are more convenient than others since they require only the knowledge of the "open-loop" transfer function and need not be in factored form. Evaluating the pole sensitivity by the residue of the closed transfer function will call for the complete determination of the closed-loop transfer function. In many cases this will not be called for in the design as interest may be restricted, for example, only in the "control poles."

The slope of the root locus is that of Δs_j (Fig. 1), but since dK/K is a positive number, one can find the angle of the slope by finding the argument of S_K^p

$$\frac{dy}{dx} = \frac{\operatorname{Im} S_K^p}{\operatorname{Re} S_K^p} = \frac{\sum \frac{A_i \sin\theta_i}{l_i}}{\sum \frac{A_i \cos\theta_i}{l_i}}$$

~ $\frac{d\omega}{d\sigma}$ in s-plan

$$= \frac{\sum A_i \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}}{\sum A_i \frac{x - x_i}{(x - x_i)^2 + (y - y_i)^2}} \tag{15}$$

where θ_i and l_i are the argument and magnitude of $s_j - s_i$ (Fig. 2). It may be easier, in some cases, to compute dy/dx from the slope of ds/dK obtained by differentiating $-K = p/q$.

Graphical Evaluation of S_K^p

S_K^p, the pole sensitivity at any value s [point A Fig. 3(b)], can be evaluated graphically by summation

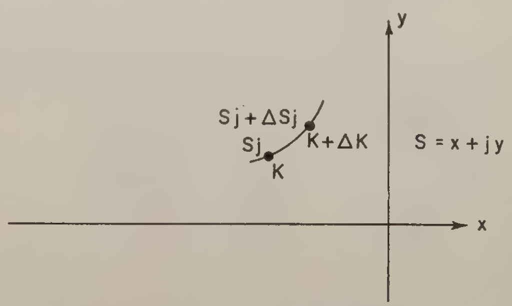


Fig. 1—Root locus slope.

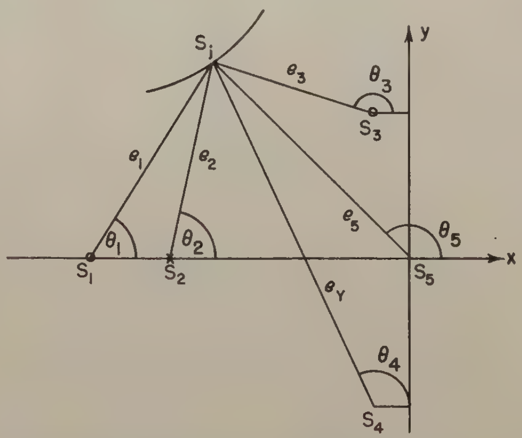
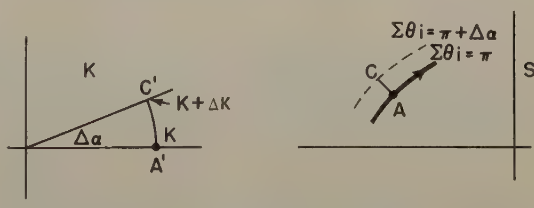


Fig. 2—Argument and magnitudes of $(s_j - s_i)$.



(a) (b)

Fig. 3—(a) Definition of ΔK. (b) Evaluation of $\Delta\alpha$.

[10] R. Y. Huang, "The sensitivity of the poles of linear closed-loop systems," *Trans. AIEE, Applications and Industry*, vol. 77, pt. 2, pp. 182–187; September, 1958. This paper came to the author's attention after the present paper had been written.

of angles and without evaluation of K, by the following procedure: Choose a point C such that AC is perpendicular to the locus and evaluate the phasor $\overline{AC} = \Delta s$. Evaluate the value of the phase of K at C by the summing angles[11] and let this angle be $\pi + \Delta\alpha$. S_K^p is then Δs over $j\Delta\alpha$.

This simple procedure is based upon the independence of the derivative of an analytic function on the direction of differentiation. Hence ΔK is chosen to be $A'C'$ which is mapped into AC. From the geometry one notes [Fig. 3(a)] that $\Delta K = j\Delta\alpha K$ and hence

$$\frac{\Delta s}{j\Delta\alpha} = \frac{\Delta s}{j\,\dfrac{\Delta K}{jK}} = \frac{\Delta s}{\dfrac{\Delta K}{K}} \doteq S_K^p. \tag{16}$$

Sensitivity to Open-Loop Poles and Zeros

Let a_i be a root of q

$$q(s) = q_i(s)(s + a_i); \tag{17}$$

then

$H(s_j, k) = q(s_j) + K\, p(s_j)$

$$\frac{\partial s_j}{\partial a_i} = -\frac{\dfrac{\partial H}{\partial a_i}}{\dfrac{\partial H}{\partial s_j}} = -\frac{q_i(s_j)}{\dfrac{\partial H}{\partial s_j}} = -\frac{q(s_j)}{\dfrac{\partial H}{\partial s_j}} \frac{1}{s_j + a_i}. \tag{18}$$

Upon comparing with (9) and (13) (a_i is a "pole"),

$$\frac{\partial s_j}{\partial a_i} = -S_K^p \frac{1}{s_j + a_i}. \tag{19}$$

Similarly if a_i is a root of p ("zero")

$$\frac{\partial s_j}{\partial a_i} = S_K^p \frac{1}{s_j + a_i}. \tag{20}$$

The Total Sensitivity

$$ds_j = \frac{\partial s_j}{\dfrac{\partial K}{K}} \frac{dK}{K} + \sum \frac{\partial s_j}{\partial a_i} da_i. \tag{21}$$

$$ds_j = S_K^p \left[\frac{dK}{K} - \sum A_i \frac{da_i}{s_j + a_i}\right]. \tag{22}$$

The Curvature of the Locus at Ordinary Points

The curvature of a curve, which is defined by means of an implicit relationship $F(x, y) = 0$ is[12]

$$\frac{1}{\rho} = \frac{F_{xx}F_y^2 - 2F_{xy}F_xF_y + F_{yy}F_x^2}{(F_x^2 + F_y^2)^{3/2}}. \tag{23}$$

The subscripts denote partial differentiation with respect to the indicated variable.

It is convenient to introduce a new Cartesian system with its origin at s_j and oriented so that the tangent to the locus coincides with the u axis (Fig. 4).

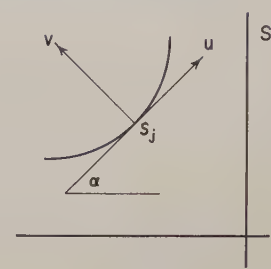


Fig. 4—u, v coordinate system.

In this system $F_u(0, 0) = 0$ and curvature is

$$\frac{1}{\rho} = -\left.\frac{F_{uu}}{F_v}\right|_{\substack{u=0\\v=0}}. \tag{24}$$

The center of curvature is on a line normal to the curve. Because of the particular choice of coordinates a positive sign of ρ means that the center of curvature lies on the positive v axis and vice versa.

K is expressed as a function of the variable $z = u + jv$, and F is defined as

$$F \triangleq \text{Im} \log K. \tag{25}$$

(Sometimes it is more convenient to define $F = \text{Im } K$.) As K takes on positive real values, F is equal to zero and therefore serves as the real function of (23).

$$\frac{1}{\rho} = -\frac{F_{uu}}{F_v} = -\frac{(\text{Im} \log K)_{uu}}{(\text{Im} \log K)_v} = -\frac{\text{Im}\,[(\log K)_{uu}]}{\text{Im}\,[(\log K)_v]}. \tag{26}$$

But for an analytic function (such as $\log K$)

$$\frac{\partial}{\partial u} = \frac{d}{dz}$$

$$\frac{\partial}{\partial v} = j\frac{\partial}{\partial jv} = j\frac{d}{dz}. \tag{27}$$

Hence

$$\frac{1}{\rho} = -\frac{\text{Im} (\log K)_{zz}}{\text{Im}\, j(\log K)_z} = -\frac{\text{Im} (\log K)_{zz}}{\text{Re} (\log K)_z} \tag{28}$$

or

[11] Truxal, *op. cit.*, p. 226.
[12] R. Courant, "Differential and Integral Calculus," Interscience Publishers, Inc., New York, N. Y., vol. 2, p. 125; 1952.

$$\frac{1}{\rho} = -\frac{\sum A_i \dfrac{\sin 2\phi_i}{l_i^2}}{\sum A_i \dfrac{\cos \phi_i}{l_i}}$$

$$= -\frac{\sum A_i \dfrac{2(u-u_i)(v-v_i)}{(u-u_i)^2+(v-v_i)^2}}{\sum A_i \dfrac{u-u_i}{(u-u_i)^2+(v-v_i)^2}} \tag{29}$$

where

$$\phi_i = \theta_i - \alpha \tag{30}$$

and α is the angle between the u axis and the x axis.

The transformation from s to z need not be carried out by evaluating (28). Consider instead the general linear transformation

$$z = (s - s_0)e^{-i\gamma}. \tag{31}$$

Then, for any function f,

$$\frac{df}{dz} = \frac{df}{ds}\frac{ds}{dz} = \frac{df}{ds}e^{i\gamma} \tag{32}$$

$$\frac{d^2f}{dz^2} = \frac{d^2f}{ds^2}\left(\frac{ds}{dz}\right)^2 + \frac{df}{ds}\frac{d^2s}{dz^2} = \frac{d^2f}{ds^2}e^{2i\gamma}. \tag{33}$$

However, since all orders of derivatives of s with respect to z higher than the first are zero,

$$\frac{d^nf}{dz^n} = \frac{d^nf}{ds^n}\left(\frac{ds}{dz}\right)^n = \frac{d^nf}{ds^n}e^{in\gamma}. \tag{34}$$

Examples are given in Appendix I.

Properties of the Root Locus at Irregular Points

The previous discussion is for the case where the locus has a definite derivative. Such points are characterized by

$$\begin{cases} F_x \neq 0 \\ F_y \neq 0 \end{cases} \quad \text{or} \quad \frac{dK}{ds} \neq 0 \neq \infty. \tag{35}$$

At points where these restrictions do not hold, the root locus has no definite derivative and a multiple point occurs. Condition (35) implies that S_K^p does not exist at an irregular point.

As an illustration, consider a point where $(\log K)'=0$ on the real axis. If both p and q have real coefficients then $\text{Im}(\log K)'=0$ everywhere on the real axis and only $\text{Re}(\log K)'=0$ need be investigated.

$$\text{Re}(\log K)' = \sum A_i \frac{x-x_i}{(x-x_i)^2+(y-y_i)^2} = 0. \tag{36}$$

If this summation is split into a term $\sum'$ corresponding to critical points on the real axis and a term $\sum''$ corresponding to critical points in the upper half plane, the following expression is obtained:

$$0 = \sum{}' A_i \frac{1}{x-x_i} + \sum{}'' A_i \frac{2(x-x_i)}{(x-x_i)^2+y_i^2}. \tag{37}$$

The factor 2 in $\sum''$ is necessary to account for the symmetry about the real axis. Eq. (37) is the well-known expression for the "breakaway" points. Another example appears in Appendix I.

The Sensitivity and Slope at Irregular Points

Let

$$K = K_0 + \frac{dK}{ds}\Delta s + \frac{1}{2}\frac{d^2K}{ds^2}\Delta s^2 + \cdots \tag{38}$$

and let the first $n-1$ derivatives vanish. Then

$$\Delta K = K - K_0 = \frac{1}{n!}\frac{d^nK}{ds^n}\Delta s^n \tag{39}$$

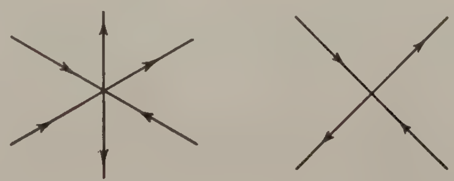
close to K_0

or

$$\Delta s = \sqrt[n]{n!\frac{\Delta K}{\dfrac{d^nK}{ds^n}}} = \sqrt[n]{\frac{n!}{\dfrac{d^nK}{ds^n}}}\sqrt[n]{\Delta K}. \tag{40}$$

As $\Delta s/\sqrt[n]{\Delta K}$ is finite it may be substituted for S_K^p at irregular points.

For any $\Delta K>0$, ds has n values separated by $2\pi/n$ which represent the outgoing branches in Fig. 5. n branches with the same separation as before are again obtained if $\Delta K<0$ but they are displaced from the previous set by $2\pi/2n$, that is, the two sets intersect (Fig. 5). Eq. (40) gives the slope at multiple points. This result is of a different form than that obtained by Papoulis.[6]



(a) (b)

Fig. 5—n outgoing branches. (a) $n=3$. (b) $n=2$.

The Curvature of One of the Locus Branches at a Double Point

A set of coordinates u, v is introduced as before (Fig. 4) at the double point with each of the branches tangent to an axis. In the vicinity of a double point, F can be factored into two functions each describing one of the branches and having no singularity.

$$F = m(u, v)M(u, v) = 0 \tag{41}$$

because of choice of coordinates

$$m(0, 0) = m_v(0, 0) = M(0, 0) = M_u(0, 0) = 0.$$

The branch tangent to the u axis is described by

$$M = \frac{F}{m} \tag{42}$$

$$M_{uu} = F\left(\frac{1}{m}\right)_{uu} + 2F_u\left(\frac{1}{m}\right)_u + F_{uu}\left(\frac{1}{m}\right). \tag{43}$$

This expression is indeterminate at $u=v=0$. After application of l'Hospital's Theorem,

$$M_{uu}(0, 0) = \frac{1}{3} F_{uu}(0, 0). \tag{44}$$

Similarly, $M_v(0, 0) = F_{uv}(0, 0)$. Hence the curvature is

$$\frac{1}{\rho} = -\frac{1}{3}\frac{F_{uuu}}{F_{uv}}. \tag{45}$$

Evaluation of the derivatives of F is carried out as before and examples appear again in Appendix I.

On Curvature, Pole Sensitivity and Their Relationship

At first glance it may seem that the curvature and sensitivity are somehow related. Example 1, Appendix I, would reveal, however, that the curvature in that example is a constant, while the pole sensitivity is (in absolute value) equal to cot ϕ—a continuous variable. The independent nature of the curvature and the pole sensitivity can be understood by noting that the pole sensitivity depends on first derivatives while the curvature depends also on the second derivative and, in general, the two derivatives are independent.

From another point of view, these two quantities are of an entirely different nature: drawing on mechanics, the curvature can be considered a "kinematic" property, while the pole sensitivity is a dynamic property.

Extension to Generalized Root Loci

Conventionally, it is assumed that K takes on only positive or negative values. This case is the one treated in detail in this paper. The results, however, can be extended easily to cases where the root locus is a mapping of some other line. Most common are the loci which are mappings either of the radii or of the concentric circles. The radii correspond to values of K with constant phase. The circles correspond to values of K for which $|K|$ = constant. The mappings of these curves are loci which correspond to constant real or constant imaginary parts of log K. F is then derived from log K and becomes

$$F = \text{Im} \log K - \gamma = 0 \tag{46a}$$

$$F = \text{Re} \log K - \log K_0 = 0. \tag{46b}$$

Since only the derivatives of F appear in the formulas derived and since the derivative of a constant is zero, all results obtained remain valid.

If K varies in such a way that either its real or imaginary parts are constant, the loci of interest are mapping of lines in Fig. 6(b). F is then derived from K itself.

The most common root locus corresponds to the real axis which is a grid line in both Fig. 6(a) and 6(b). Hence, F can be derived for this case from either K or log K as is done in the examples.

Sensitivity Caused by Large Variations in Parameters

Consider the transmission T of the graph shown in Fig. 7.

$$T = \frac{acK}{1 - Kb} = T(s, k). \tag{47}$$

Let $K = K_0 + \Delta K$ and $T(K_0) = T_0$. Then,

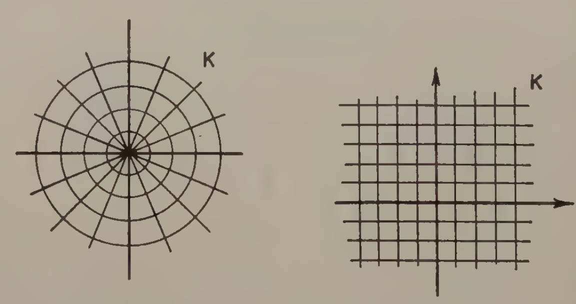
$$\frac{T}{T_0} = \frac{1}{1 - \dfrac{\dfrac{\Delta K}{K}}{1 - K_0 b}}. \tag{48}$$

But,

$$\frac{1}{1 - K_0 b} = S_K^T = S. \tag{49}$$

Hence,

$$T = T_0 \sum_0^\infty \left(\frac{\Delta K}{K} S\right)^n \tag{50}$$



(a) (b)

Fig. 6—(a) Loci for K of constant magnitude and phase. (b) Loci for K with constant real and imaginary parts.

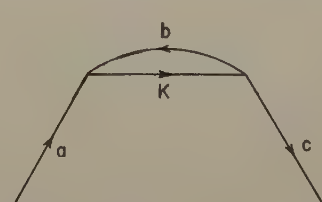


Fig. 7—Typical flow graph with one feedback loop, no direct transmission.

or

$$\ln \frac{T}{T_0} = \sum_{1}^{\infty} \frac{1}{n}\left(\frac{\Delta K}{K} S\right)^n. \quad (51)$$

These formulas are interesting since they show that as a result of the special form of T, the sensitivity determines all coefficients of the power series expansion of T.

The Root Locus When One of the Critical Points Becomes Variable

In many cases, the position of one of the critical points (a pole or a zero) is either not known exactly or is a variable of design. Thus, it may be of advantage to know the locus of the poles when this movement is considered. With notations similar to that of (17), one can show that the locus is described by the roots of

$$q(s) + Kp_i(s)(s + a_i + \Delta a_i) = q(s) + Kp(s) + \Delta a_i K p_i(s). \quad (52)$$

K here is considered constant and Δa_i, the variable. Eq. (52) indicates that the new poles are the poles of the closed-loop system while the zeros are those of p_i—in other words, the zeros of the original system less the movable zero. As an illustration, consider the second-order system of Fig. 8. The loop is closed and a certain gain of value K is chosen so that the closed-loop poles have the positions A and B. As there was only one zero in the original system, the a locus is that of a system with two poles at A and B and no zeros. The locus is indicated by the dashed line and the real axis in Fig. 8. A similar rule applies to a variable open-loop pole.

As another example, consider a commonly used transfer function [Fig. 9(a)]

$$\frac{K}{s(s+a)} \quad (53)$$

around which the loop is closed and the selected gain K corresponds to two closed-loop poles at A and B. The position of the closed loop poles for different values of a is given according to the rules by the locus shown in Fig. 9(b) by a dashed line. According to the rules given before, there are "poles" at A and B and there is a zero at C, and the "gain" is $\pm\Delta a$.

The Locus When Two Conjugate Poles Move

One has now to consider the roots of

$$(s^2 + b_i s + \Delta b_i s + c_i + \Delta c_i)q_i(s) + Kp(s) = q(s) + Kp(s) + \Delta b_i s q_i(s) + \Delta c_i q_i(s) = 0. \quad (54)$$

Two cases will be considered:

1) The two poles move on a vertical line and remain conjugate.[13] In this case, $\Delta b_i = 0$ and the locus has "poles" at the poles of the closed loop and the "zeros" are the open-loop poles which do not move.

2) The two conjugate poles move on a circle centered at the origin, as in Fig. 10. This assumes that the product of their values is constant and hence $\Delta c_i = 0$, only Δb_i varies. Examination of (54) shows that while the "poles" are the same as before, the zeros are given by q_i and, in addition, there is a zero at the origin. By change of variable, it is easy to prove that if the variation is on a circle centered anywhere on the real axis, not just at the origin, the added zero has to be located at its center.

[13] This requirement is equivalent to stating that the sum of the two poles remains constant.

Equivalent Loci

Theorem: The root locus remains unchanged, if, instead of the original poles, one uses any set of points on the locus corresponding to any desired $K = K_1$. In this case, K has to be replaced by $K - K_1$.

Proof: The root locus is characterized by the zeros of (9) but

$$q + Kp = (q + K_1 p) + (K - K_1)p = 0. \quad (55)$$

The right member describes the new locus mentioned in the theorem.

Theorem: Interchange of open-loop poles and zeros would create an identical locus, which, however, has an opposite sense and in which K is replaced by $1/K$. In

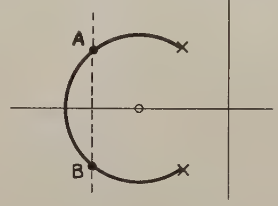
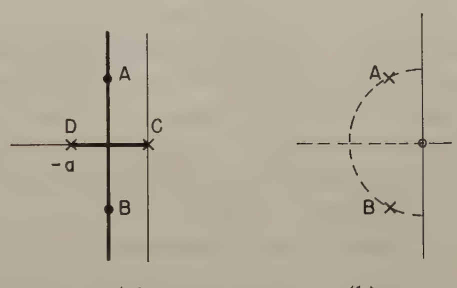


Fig. 8—Loci for a second-order system with a variable zero.



(a) (b)

Fig. 9—(a) Root locus for system with two real poles. (b) Position of poles for various values of a.

Fig. 10—Locus change for variation of conjugate poles on a circle.

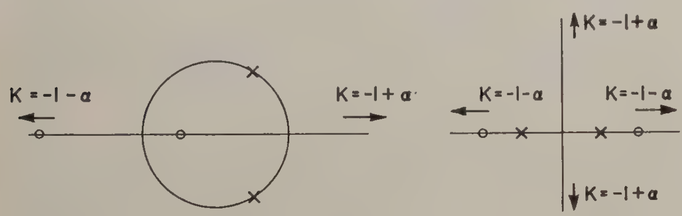
the normal case of excess of poles over zeros, the loci approach the asymptotes when K (the new one) is very small. For proof, divide (9) throughout by K.

Loci with No Open-Loop Poles or Zeros at Infinity

If the order of $p(s)$ and $q(s)$ is the same, then for $K=-1$, the order of (9) would be reduced (assuming unity to be the leading coefficient in p and q). Hence some roots will disappear or become points at infinity. Each point at infinity has one entering branch and one outgoing branch. The statement follows immediately, if in (9) $-1+\alpha$ is substituted for K:

$$H(s, \alpha) = (q - p) + \alpha p = 0. \tag{56}$$

Here p is of higher order than $q-p$ and hence the points at infinity occur for $\alpha \to 0$. Let the difference in order be n. There would be n asymptotes as $\alpha \to +0$, and another set of n asymptotes as $\alpha \to -0$. The two sets intersect each other. (This is the same as in Fig. 5, except that the multiple point is now at infinity.) The rules for centroid, number of asymptotes, etc., are the usual ones, with $q(s)-p(s)$ replacing q. Fig. 11 describes two such cases.



(a) (b)

Fig. 11—Root loci for a system with an equal number of poles and zeros.

Conclusion

The usefulness of any analytical technique depends, to a large extent, on the familiarity of the user with that technique. This familiarity readily enables the user to extract more information about the properties of his problem, and guides him to the correct path of analysis without time-consuming false starts. Such familiarity requires the understanding of general theorems applicable to all problems where the technique applies. It is felt, therefore, that the utility of root locus analysis will be enhanced by the results given in this paper. These results can be classified in several ways: properties in the "large" and "small." The properties in the large which appear in the latter part of the paper are directed toward some particular cases. The case when there are no poles or zeros at infinity may occur quite often, yet the basic known textbook rules regarding asymptotes would be misleading here. The ability to obtain the critical points of a system in which an open-loop pole or pole-pair is varying is felt to be a useful design tool. The properties in the small, such as slope, curvature, and location of multiple points can be very helpful in drawing the locus as in Example 2, Appendix I.

The rest of the paper considers sensitivity relations which should be of particular value in design of practical systems where elements are either unknown or variable, and these tolerances have to be considered in trying to meet specifications. These ideas, and the deliberate variation of elements to vary properties of systems in a desired way to make them adaptive, can be made a basis for new design procedures.

Appendix I

Example 1 (Fig. 12)

$$K = -\frac{s^2+1}{s} = -s - \frac{1}{s} \tag{57}$$

$$\frac{dK}{ds} = -1 + \frac{1}{s^2}; \quad \frac{d^2K}{ds^2} = -\frac{2}{s^3}$$

$$\frac{d^3K}{ds^3} = \frac{6}{s^4}. \tag{58}$$

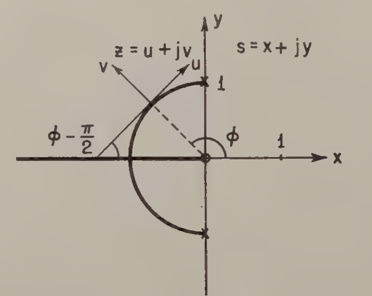


Fig. 12—System discussed in Example 1.

Double points occur at $dK/ds=0$ or $s=\pm 1$.

$$\frac{dK}{ds} = -2je^{-i\phi}\sin\phi = 2\sin e^{-i(\phi+\pi/2)} \tag{59}$$

for $K<2$ or on the circular portion of the root locus.

$$\frac{dy}{dx} = -\frac{\operatorname{Im}\dfrac{dK}{ds}}{\operatorname{Re}\dfrac{dK}{ds}} = -\cot\phi = \tan\left(\phi - \frac{\pi}{2}\right). \tag{60}$$

The pole sensitivity will be

$$S_K^p = K\frac{ds}{dK} = -\frac{s^2+1}{s}\,\frac{s^2}{1-s^2} = s\,\frac{s^2+1}{s^2-1}$$

$$= s\,\frac{s+\dfrac{1}{s}}{s-\dfrac{1}{s}} \tag{61}$$

and on the circular portion

$$S_K^p = e^{j\phi} \frac{e^{j\phi} + e^{-j\phi}}{e^{j\phi} - e^{j\phi}} = -je^{j\phi} \cot \phi. \tag{62}$$

The curvature is, using (34),

$$\frac{1}{\rho} = -\frac{F_{uu}}{F_v} = -\frac{\operatorname{Im}(-2e^{-3j\phi})e^{j(2\phi-\pi)}}{\operatorname{Im}(-j2\sin\phi e^{-j\phi}e^{j(\phi-\pi/2)})} = -1. \tag{63}$$

The curvature at $s=-1(\phi=\pi)$ is

$$\frac{1}{\rho} = -\frac{\frac{1}{3}\operatorname{Im}\frac{d^3K}{dz^3}}{\operatorname{Im} j\frac{d^2K}{dz^2}} = -\frac{1}{3}\frac{\operatorname{Im}\frac{6}{(-1)^4}e^{j3(\pi/2)}}{\operatorname{Im}\left(j\frac{-2}{(-1)^3}\right)e^{j2(\pi/2)}} = -1\cdot \tag{64}$$

The pole sensitivity at $s=-1$ (a double point) is infinite and hence (40) is evaluated. Here $n=2$ and, according to (40),

$$\frac{\Delta s}{\sqrt{\Delta K}} = \sqrt{-\frac{2!}{\frac{2}{s^3}}} = \pm 1. \tag{65}$$

All these values can be verified otherwise.

The graphical procedure for obtaining $S_K{}^p$ is tried at $\phi=135°$(Fig. 13). AC is chosen to be 0.20 and one finds that

$$\sum \theta_i = \pi - 12\frac{\pi}{180} = \pi - 0.21. \tag{66}$$

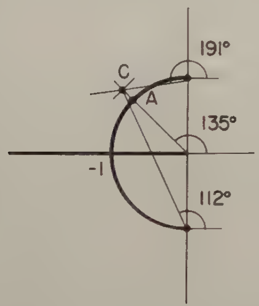


Fig. 13—Graphical procedure for finding S_K^p.

Hence

$$S_K{}^p = \frac{\Delta s}{j\Delta\alpha} = -\frac{-0.20e^{0.75\pi j}}{j0.21} = 0.95je^{j0.75\pi}. \tag{67}$$

According to (62) the exact value is 1 instead of 0.95.

Example 2 (Fig. 14)

$$K = \frac{1}{(s+2)(s-2)(s^2+16)}\cdot \tag{68}$$

It will be more convenient to consider $u=-1/K$ which corresponds to the same root locus.

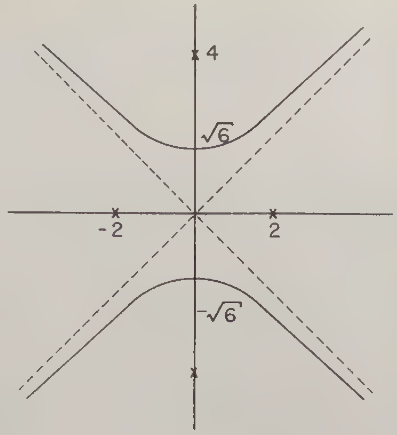


Fig. 14—System discussed in Example 2.

$$u = s^4 + 12s^2 - 64 \tag{69}$$

$$\frac{du}{ds} = 4s^3 + 24s = 4s(s^2+6)$$

$$\frac{d^2u}{ds^2} = 12s^2 + 24$$

$$\frac{d^3u}{ds^3} = 24s. \tag{70}$$

Multiple points occur at $du/ds=0$ which yield $s=0$, $s=\pm j\sqrt{6}$. These values of s correspond to real values of u and hence are double points of the root locus. The curvature at $s=j\sqrt{6}$ is

$$\frac{1}{\rho} = -\frac{1}{3}\frac{\operatorname{Im} 24j6}{\operatorname{Im} j(-12\times 6+24)} = \frac{1}{\sqrt{6}}\cdot \tag{71}$$

$ds/dz=1$ in this case.

Appendix II

On the Meaning of Sensitivity Functions

The accepted definition of sensitivity utilizing logarithmic derivatives might be ambiguous in some cases, especially when T is small or zero. For a balanced bridge circuit, the sensitivity (with respect to any element) of E_0/E_{in} is infinite while an infinitesimal change in an element value corresponds to an infinitesimal change in T or E.

Another example involves the sensitivity of a circuit to a resistance R. R is composed as a series combination of two resistances, R_1 and R_2, where $R_1=10R_2$. Then $S_{R_2}^T=11S_R^T$, while an ordinary derivative would make the total sensitivity equal to that of the sum.

Acknowledgment

The author wishes to acknowledge with gratitude the advice and suggestions of Prof. J. G. Truxal of the Polytechnic Institute of Brooklyn, Brooklyn, N. Y. and Prof. J. R. Clark, formerly of Purdue University, Lafayette, Ind.

Time Lag Systems—A Bibliography*

N. H. CHOKSY†

Summary—In a recent issue of these TRANSACTIONS, Weiss[1] has given an excellent annotated bibliography on the subject of transportation lag. He was kind enough to refer to a previous bibliography which appeared in this author's thesis (his reference number [Ch5]). Since the latter is not generally available, the bibliography therein is given here. The items which have already appeared in Weiss' listing have been omitted; items which have appeared since the thesis was written have been included.

While most of the items here were found by library searches, acknowledgment must be made to Bellman's[2] bibliographies as a source.

If there are items which have not been mentioned either here or in Weiss' paper, the author of the present paper would appreciate being informed about it by either a complete reference or a copy of the paper (or papers) in question.

A short introduction on the mathematical characterization of time lag systems is given before the bibliography.

Introduction

SINCE transient analysis is often of interest, we shall consider here only the homogeneous forms of the governing equations.

Linear physical systems[3] are characterized by linear differential equations or by a set of such equations (and, as is well known, one nth order differential equation can always be written as a set of n first-order differential equations). Here, for convenience, we shall consider one equation. This, in general, has the form:

$$a_0 y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) \equiv \sum_{i=0}^{n} a_i y^{(n-i)}(t) = 0 \quad (1)$$

where the a_i are real or complex-valued constants, or may be functions of time.

Associated with such an equation there is the so-called characteristic equation which is a polynomial equation in a complex variable if the a_i are constants, or, indeed, may itself be a differential equation if the a_i are polynomial functions of time. Again, here we shall restrict ourselves to the former case. Then, the characteristic equation can be written as:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_n \equiv \sum_{i=0}^{n} a_i s^{n-i} = 0. \quad (2)$$

The study of the behavior of the physical system by an analysis of this equation fills the literature.

Now, a differential equation as expressed in (1) tacitly implies that the future behavior of the system is fully determined if we have complete knowledge of its state, *i.e.*, of the values of $y^{(i)}(t)$, $(i=0, 1, \cdots, n)$ at any given time t. In many systems, particularly in the fields of economics, population studies, biophysics, physiology, etc., this implication is not justifiable. In such systems, the future behavior is dependent not only on its present state, but also in some measure on its immediate past, and, indeed, in some cases on its immediate future.

The author has always preferred to use the term "transportation lag" only when the dependency on the past (or future) is through the dependent variable only and *not* through any of its derivatives. The words "time lag" are reserved for the more general case.

In the transportation lag case, if no derivatives are present, the corresponding equation becomes a difference equation. However, of more interest is the time lag case where the corresponding equation is called a *difference-differential equation*. Systems characterized by such equations are variously called time-lag systems, retarded systems, hereditary systems, etc. These equations have the form:

$$\sum_{i=0}^{n} a_i y^{(n-i)}(t) + \sum_{k=0}^{m} b_k y^{(m-k)}(t - \tau_k) = 0 \quad (3)$$

where a_i and b_k are constants, τ_k are real constants (positive for dependency on the past, negative for dependency on the future) or may, in complicated cases, be functions of time (we shall not consider this latter possibility here), and m may be greater than, equal to, or less than n. If for a certain k, $\tau_k = 0$, then the corresponding $b_k \equiv 0$.

The characteristic equation in such cases is no longer a polynomial equation, but a transcendental equation of the sort usually termed exponential polynomial equations:

$$\sum_{i=0}^{n} a_i s^{n-i} + \sum_{k=0}^{m} b_k e^{\tau_k s} s^{m-k} = 0. \quad (4)$$

This equation plays a part, analogous to that of (2), in the study of time lag systems.

This brief introduction can be closed by simply stating that instead of difference-differential equations, time lag systems are also characterized by integral equations of the form:

$$y(t) = f(t) + \int_0^t K(u) y(t-u) du. \quad (5)$$

This is called the renewal equation.

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[1] R. Weiss, "Transportation lag—an annotated bibliography," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-4, pp. 56–68; May, 1959.

[2] *Ibid.*, references [Be2], [Be3], [Be4].

[3] The word "physical" is used here in the widest possible sense to include all phenomena in the real world which may be characterized mathematically.

The following bibliography lists papers concerning time lag systems in a variety of ways: some papers discuss physical systems and show that a difference-differential equation characterizes them; some go further and try to obtain solutions to such equations directly in the time domain; some examine the solutions for asymptotic behavior; some deal with the stability of the systems through the study of their characteristic exponential polynomial equations, and so on. As the titles indicate, a large assortment of problems are considered, in the hope of covering all possible fields of interest.

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Addendum

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Correspondence

Properties of Root Locus Asymptotes*

As a function of K, the roots of the rational polynomial $P(s)+KZ(s)$ describe what are known as root loci in the complex s plane. The behavior of these roots is important in understanding the behavior of linear systems as a function of the variable K. The variable K may be either the gain of some part of the system or some control parameter in the system.

Based on the location of the roots of $P(s)$ and $Z(s)$ in the s plane, the root loci of $P(s)+KZ(s)$ can be sketched by applying a number of standard rules.[1] A new rule is presented here which indicates the final side from which the locus approaches its asymptote as K and $|s|$ become very large.

The roots of

$$P(s) = \prod_{i=1}^{n} (s - s_{p_i})$$

are known as poles, whereas the roots of

$$Z(s) = \prod_{i=1}^{m} (s - s_{z_i})$$

are known as zeros. It is well known that, as K goes from zero to large positive numbers, the roots proceed along the loci from the poles to the zeros. If there are more poles than zeros ($n>m$) in the finite s plane, the loci approach ($n-m$) asymptotes radiating from the centroid of the finite pole-zero pattern. The centroid is located at

$$c = \frac{\sum_{i=1}^{n} s_{p_i} - \sum_{i=1}^{m} s_{z_i}}{n - m}, \tag{1}$$

and the angles of the asymptotes are located at

$$\phi_k = \frac{2k+1}{n-m}\pi; \quad k = 0, 1, \cdots, n-m-1. \tag{2}$$

The side from which the locus approaches the asymptote is determined as follows:

1) The locus lies exactly along the asymptote if the pole-zero pattern is symmetric about the asymptote line extended through the centroid. An analytical test for this is the substitution of

$$s = s'e^{j\phi_k} \tag{3}$$

into the polynomials $P(s)$ and $Z(s)$. The locus lies along the asymptote if, and only if, the coefficients of the new polynomials in s' are real. If this test fails,

2) Find the rational polynomial of $P(s)$ and $Z(s)$ in the form

$$Z(s) = (s-c)^m + a_1(s-c)^{m-1} + \cdots + a_m$$

$$P(s) = (s-c)^n + b_1(s-c)^{n-1} + \cdots + b_n \tag{4}$$

where c is the centroid.

The coefficients a_q, b_q may be found directly from the pole-zero configuration by

$$a_q = (-1)^q \sum_{i=1}^{m-q+1} \sum_{j=i+1}^{m-q+2} \cdots \sum_{k=j+1}^{m} \cdot (s_{z_i} - c)(s_{z_j} - c) \cdots (s_{z_k} - c). \tag{5}$$

An identical expression gives b_q, except that b_q uses the poles where (5) has zeros. These coefficients may be found from the coefficients of s rather than of $(s-c)$ by

$$a_q = \sum_{i=0}^{q} \binom{m-p+i}{m-p} k_{p-i}c^i, \tag{6}$$

where k_j is the coefficient of s^{m-j} in $Z(s)$. The expression for b_q is the same, with k_j replaced by the coefficients of s^{n-j} in $P(s)$.

3) Determine the first nonzero number

$$d_p = b_p - a_p; \quad p = 1, 2, 3 \cdots \tag{7}$$

4) Divide the s plane into $2p$ equal sectors about the centroid starting with the positive real-axis, as in Fig. 1.

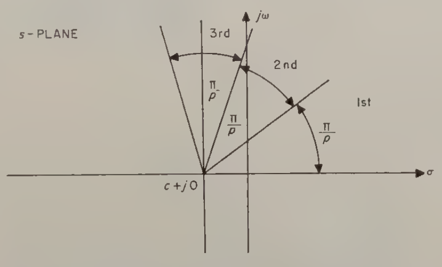

Fig. 1—Division of the s plane into $2p$ sectors. If asymptote lies in odd sector: asymptote approaches from right, if $d_p<0$; from left, if $d_p>0$. If asymptote lies in even sector: asymptote approaches from left, if $d_p<0$; from right, if $d_p>0$

5) If the asymptote lies in an odd sector, the locus approaches from the right if $d_p<0$ and from the left if $d_p>0$. If the asymptote lies in an even sector, the opposite is true: the locus approaches from the right if $d_p>0$ and from the left if $d_p<0$.

6) If the asymptote falls on a sector line, determine the nonzero number

$$d_q = b_q - a_q - \sum_{i=p}^{q-1} d_i a_{q-i} \tag{8}$$

of least q such that the asymptote does not lie on one of the new lines which divide the s plane into $2q$ sectors. Apply steps 4) and 5), with p replaced by q.

In all cases, d_1 will be zero. The constants a_2 and b_2 can be obtained either from the polynomial or from the pole-zero pattern by use of the formula

$$a_2 = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \alpha_{z_i}\alpha_{z_j} + \frac{1}{2}\sum_{i=1}^{m} \beta_{z_i}^2, \tag{9}$$

where

$$\alpha_{z_i} = \mathrm{Re}\,(s_{z_i} - c)$$

$$\beta_{z_i} = \mathrm{Im}\,(s_{z_i}).$$

The formula for b_2 is identical to that for a_2, except that the b_2 formula uses the poles. Similar expressions exist for a_p and b_p for p greater than 2. The numbers a_2 and b_2 may also be obtained from the coefficients of s rather than $(s-c)$:

$$a_2 = k_2 + (m-1)k_1c + m(m-1)c^2, \tag{10}$$

where k_j is the coefficient of s^{m-j} of $Z(s)$ and c is the centroid. An identical expression exists for b_2, except that k_j is the coefficient of s^{n-j} of $P(s)$.

Two examples are given to illustrate the use of the asymptote rule:

1) Consider the polynomial

$$P(s) + KZ(s) = s^4 + 7s^3 + 12s^2 + K(s+1) = s^2(s+4)(s+3) + K(s+1).$$

The root locus of this polynomial has a centroid at

$$c = \frac{(-4-3)-(-1)}{4-1} = -2 + j0.$$

There are three asymptotes at angles of 60°, 180°, and 300°, as shown in Fig. 2. From the polynomial written in the form

$$(s+2)^4 - (s+2)^3 - 6(s+2)^2 + 4(s+2) + 8 + K[(s+2) - 1]$$

it is determined that

$$a_2 = 0$$

$$b_2 = -6.$$

Since $b_2<a_2$ the locus approaches the 60° asymptote from the right and the 300° asymptote from the left. An identical result is obtained for a_2 and b_2 by using (9) directly on the pole-zero pattern.

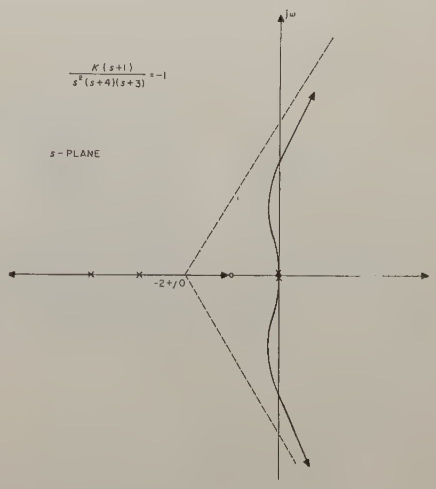

Fig. 2—Root locus for Example 1.

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[1] J. F. Truxal, "Control System Synthesis," ch. 4, "Root-Locus Methods," McGraw-Hill Book Co., Inc., New York, N. Y.; 1955.

2) Consider the polynomial

$$
\begin{aligned}
P(s) &+ KZ(s) \\
&= s^5 + 4s^4 + 6s^3 + 4s^2 + K(s^2 + 4s + 8) \\
&= s^2(s^2 + 2s + 2)(s + 2) + K(s^2 + 4s + 8).
\end{aligned}
$$

The root locus of this polynomial has a centroid at the origin and three asymptotes at 60°, 180°, and 300°, as shown in Fig. 3.

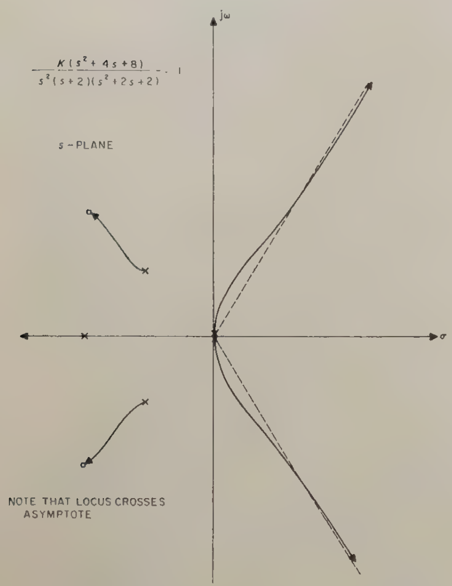

Fig. 3—Root locus for Example 2. Note that locus crosses asymptote.

From this polynomial,

$$a_2 = 8$$
$$b_2 = 6.$$

The 60° asymptote is approached from the right since $a_2 > b_2$. Note that the locus crosses the asymptote, a fact which might not be evident without an actual detailed plot of the root locus.

The asymptotic behavior of the root locus is obtained by locating the root of the polynomial $P(s)+KZ(s)$ for large values of K and $|s|$. Equating the polynomial $P(s)+KZ(s)$ to zero produces the relation that

$$
\begin{aligned}
-K = \frac{P(s)}{Z(s)} &= (s - c)^{n-m} + d_1(s - c)^{n-m-1} \\
&\quad + d_2(s - c)^{n-m-2} + \cdots \qquad (11)
\end{aligned}
$$

The coefficients d_q are

$$
\begin{aligned}
d_1 &= b_1 - a_1 = 0 \\
d_q &= b_q - a_q - \sum_{i=1}^{q-1} a_i d_{q-i}, \qquad (12)
\end{aligned}
$$

where $b_q = 0$ for $q > n$, $a_q = 0$ for $q > m$.

From the identity

$$(s - c)^{n-m} = -K\left[\frac{-K}{(s - c)^{n-m}}\right]^{-1} \qquad (13)$$

extraction of the $(n-m)$th root, and transposition of the c term to the right-hand side, the following equation is obtained:

$$
s = c + K^{1/(n-m)} e^{j[(2k+1)/(n-m)]\pi} \cdot \left[\frac{-K}{(s - c)^{n-m}}\right]^{-1/(n-m)}. \qquad (14)
$$

Substitution of (11) into (14) and application of the binomial theorem allow s to be written

$$
\begin{aligned}
s = c + K^{1/(n-m)} e^{j[(2k+1/(n-m)]\pi} &\left\{1 - \frac{1}{(n - m)}\right. \\
&\cdot [d_1(s - c)^{-1} + d_2(s - c)^{-2} + \cdots] \\
&+ \frac{1}{2!}\frac{(1 + n - m)}{(n - m)^2} [d_1(s - c)^{-1} \\
&\left. + d_2(s - c)^{-2} \cdots]^2 + \cdots\right\}. \qquad (15)
\end{aligned}
$$

For very large $|s-c|$, (15) shows that

$$K \approx -(s - c)^{n-m}. \qquad (16)$$

Under such conditions, (15) may be approximated by keeping only the first two terms within the braces. For large $|s-c|$, the inner brackets become $d_p(s-c)^{-p}$, where d_p is the first nonzero coefficient. Then

$$
\begin{aligned}
s &= c + K^{1/(n-m)} e^{j[(2k+1)/(n-m)]\pi} \\
&\quad - \frac{d_p}{n - m} {}^{(1-p)/(n-m)} e^{j[(2k+1)/(n-m)]\pi(1-p)}. \qquad (17)
\end{aligned}
$$

Note that the first two terms of (17) constitute the vector from the origin to points along the asymptotes, and the third term is a first-order correction term possibly displacing the root off the asymptote.

The asymptote vector is

$$c + K^{1/(n-m)} e^{j[(2k+1)/(n-m)]\pi} = c + \rho e^{j\phi_k}, \qquad (18a)$$

while the first-order correction is

$$\frac{1}{n - m} K^{(1-p)/(n-m)} e^{+j[(2k+1)/(n-m)]\pi(1-p)} = \lambda e^{j\phi_k(1-p)}, \qquad (18b)$$

where ϕ_k is the kth asymptote angle. This approximation for the root of the polynomial represents the offset due to the centroid, the asymptote, and the lowest-order correction to the asymptote. Note that there are $(n-m)$ asymptotes at angles $\pi(2k+1)/(n-m)$, that the asymptote grows as $K^{1/(n-m)}$, and that the correction decreases as $K^{(1-p)/(n-m)}$. The side from which the locus approaches the asymptote can be determined by considering the angle of the correction in relation to the angle of the asymptote. Substitution of (18a) and (18b) into (17) gives

$$s = c + \rho e^{j\phi_k} - \lambda d_p e^{j\phi_k(1-p)}. \qquad (19)$$

Divide the complex plane into $2p$ sectors by lines through the centroid c at angles of $r\pi/p$, and consider points about an asymptote. If ϕ_k is an odd sector, that is,

$$\frac{2r\pi}{p} < \phi_k < \left(\frac{2r + 1}{p}\right)\pi,$$

then

$$\phi_k - 2r\pi > \phi_k(1 - p) > \phi_k - (2r + 1)\pi. \qquad (20)$$

Obviously, in these regions, for $-\lambda d_p > 0$ the locus must lie to the right of the asymptote. If $-\lambda d_p < 0$, $\phi_k(1-p)$ is increased by π radians and the locus lies to the left. Examination shows that this condition reverses at multiples of π/p radians. If equal sectors of angle π/p are constructed in the s plane about the centroid, the angle condition is equivalent to determining the sector in which the asymptote lies. This argument then produces the first part of the rule stated here.

If the lowest-order correction lies along the asymptote, it will not give an indication of the approach side. Likewise, powers of this correction term will not produce an indication. This case occurs when the asymptote being considered lies on a sector line.

The next lowest-order correction results from the next nonzero d_p. The angle condition requires a new set of sectors to be constructed in the s plane. If the asymptote still lies on a sector line, the next nonzero d_p must be considered, and so forth, until one is found which produces a sector which includes the asymptote. Products and cross-products of the lower-order d_p's and the one under consideration produce either corrections along the asymptote or higher-order corrections than the one being considered. Thus, these powers and cross-powers may be neglected.

If all nonzero d_i $(i = p \cdots \infty)$ require sector angles which lie along a certain asymptote, then from (15) there can be no displacement from any term, and thus the locus must *coincide* with the asymptote. This condition is fulfilled if the pole-zero pattern is symmetric about the asymptote line through the centroid. This then completes the rest of the rule.

Most pole-zero patterns are such that the loci in the upper half-plane all converge to the asymptotes from the upper side or all from the lower side. This case occurs when the coefficients a_2 and b_2 are nonequal.

The coefficients a_2 and b_2 can be determined from $Z(s)$ and $P(s)$ expanded about the centroid c:

$$
\begin{aligned}
Z(s - c) &= (s - c)^m + a_1(s - c)^{m-1} \\
&\quad + a_2(s - c)^{m-2} + \cdots \\
P(s - c) &= (s - c)^n + b_1(s - c)^{n-1} \\
&\quad + b_2(s - c)^{n-2} + \cdots
\end{aligned}
$$

or from the pole-zero pattern with the use of (9).

In such cases, these coefficients a_2 and b_2 determine the approach to all asymptotes other than those at ±90°. For those asymptotes not equal to ±90°, the loci in the upper half-plane converge from the upper side of the asymptote if $b_2 > a_2$ and from the lower side of the asymptote if $b_2 < a_2$.

A method of determining the final side to which the root locus converges to its asymptote has been presented. It is possible to make this determination analytically from the polynomial or visually from the pole-zero pattern.

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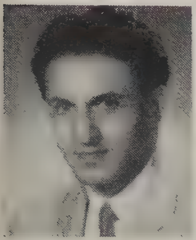
Contributors



R. Bellman

Richard Bellman was born on August 26, 1920, in New York, N. Y. He received the B.A. degree, from Brooklyn College, Brooklyn, N. Y., in 1941 and a Ph.D. degree from Princeton University, Princeton, N. J., in 1946, both in mathematics.

He has held the following academic positions: assistant professor of mathematics at Princeton University, 1946 to 1948; associate professor of mathematics at Stanford University, Stanford, Calif., 1948 to 1952; and visiting Professor of Engineering at the University of California, Los Angeles, in 1956.

At the present Dr. Bellman is a mathematician at The RAND Corporation in Santa Monica, Calif., and a consultant for the AEC, BOOZ, and Hughes Aircraft Co.

Dr. Bellman has also been active in wartime and defense activities as a USAF instructor in a preradar course (1942–1943), as an instructor in Specialized Training Program (1943–1944), as a mathematician in a sonar project at the U. S. Naval Radio and Sound Laboratory, Point Loma, San Diego (1944), as a mathematician on the atomic bomb project in the special engineering division, U. S. Army, Los Alamos, N. M. (1944–1946), and as a member of Project Matterhorn, which was concerned with thermonuclear weapons, at Princeton University, 1951–1952.

He has published three books and over two hundred and fifty papers.

Dr. Bellman is a member of the Council of the American Mathematical Society, a member of the Committee on Applied Mathematics of the American Mathematical Society, and was chairman of the Society for Industrial and Applied Mathematics, Los Angeles Chapter, 1956.

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N. T. Bold

Norbert T. Bold (S'55) was born in Chicago, Ill., on September 3, 1929. He received the B.E.E. degree from Marquette University, Milwaukee, Wis., in 1951 and the M.S. and Ph.D. degrees from Northwestern University, Evanston, Ill., in 1956 and 1958, respectively.

From 1951 to 1954, he served in the U. S. Navy as engineering officer aboard an aircraft carrier in the Korean Theater of Operations and a minesweeper in the Atlantic Fleet.

While at Northwestern University, he was employed as a graduate teaching assistant in the electrical engineering department for three years. The last year of his education was completed with the assistance of a James M. Barker fellowship.

Dr. Bold has been a student member of the AIEE. He is also a member of Tau Beta Pi, Eta Kappa Nu, Pi Mu Epsilon, and Sigma Xi.

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H. Chestnut

H. Chestnut (SM'57) was born on November 25, 1917 in New York, N. Y. He received the B.S.E.E. and M.S.E.E. degrees in 1939 and 1940, respectively, from the Massachusetts Institute of Technology, Cambridge.

He has been with the General Electric Company since 1940, when he joined the Test Course. In 1942, he supervised the G.E. Electrical B Advanced Engineering Program. Upon completion of this program in 1943, he joined the Aeronautics and Ordnance Systems Division. From 1943 to 1956, he had assignments in AOS in Systems Development, Advanced Development, Guided Missiles, and Marine Systems sections. In 1954, he was appointed Ordnance Engineer, and in 1955, Project Engineer F-104. Since 1956, he has been Control Systems Engineer in the General Engineering Laboratory of General Electric, Schenectady, N. Y.

Mr. Chestnut has been guest instructor in the Extension Division of Union College, Schenectady, a member of the Research and Development Board Committees, the chairman of the AIEE Feedback Control Systems Committee and the AIEE Subcommittee on Component Specifications, Vice-Chairman of the North American Control Council, and the first President of the International Federation of Automatic Control.

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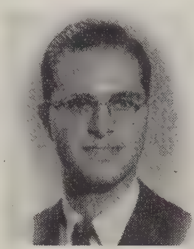
Nasli H. Choksy (A'50–M'55), for a photograph and biography, please see page 117 of the December, 1958, issue of these Transactions.



L. A. Gimpelson

Lester A. Gimpelson (S'55) was born on July 15, 1935, in New York, N. Y. In 1959, he received the S.B. and S.M. degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, where he is continuing his graduate study.

While on M.I.T.'s Cooperative Course program he was employed by the Air Force Cambridge Research Center, Bedford, Mass., where he was engaged in the design of electronic and transistor circuits. He was also associated with the Research Laboratories of the British Thomson-Houston Company at Rugby, England. Since 1957 he has been a Staff member at M.I.T., first as a research assistant, and at present as a teaching assistant.

Mr. Gimpelson is a member of Tau Beta Pi, Eta Kappa Nu, and Sigma Xi.

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W. A. Janos

William A. Janos (M'59) was born on November 9, 1926, in Easton, Pa. He served in the U. S. Army from 1945 to 1947, and he graduated from Rutgers University in 1951 with a B.S. degree in physics. In 1954 he obtained his M.A. degree in physics, and in 1958 he received a Ph.D. in physics, both from the University of California, Berkeley. During this time he was the recipient of a University appointed teaching assistantship in the Physics Department and a Convair Scholarship Award.

He has been employed by Convair and Convair Astronautics, San Diego, Calif., from 1951 until the present except for leaves of absence to obtain his advanced degrees. His work has covered a variety of fields including applied analysis and spectral theory related to analytical dynamics, wave propagation and diffraction, variational techniques in least time trajectories for thrust propelled flight, control system analysis and synthesis, noise theory and optimal linear estimation.

He is presently conducting analytical research in digital-analog communication, computation and control systems, and initiating studies in information theory of continuous sources.

Dr. Janos is a member of the American Physical Society.

Robert Kalaba was born on September 21, 1926 in Mt. Vernon, N. Y. He received both a B.A. in February, 1948 and a Ph.D. in February, 1958 from New York University, New York, N. Y.

R. Kalaba

He served in the U. S. Navy during 1945–1946 and was a teaching fellow in mathematics at N. Y. U. from 1948 to 1951. Since 1951 he has been a mathematician in the Electronics Department of The RAND Corporation, Santa Monica, Calif. Over forty of his published papers have dealt with mathematical aspects of a variety of problems in physics, engineering, and operations research. Wave propagation and automatic control are among his current interests.

He is a lecturer in engineering at University of California at Los Angeles, and a member of Phi Beta Kappa, the American Mathematical Society, the Mathematical Association of America and the Society for Industrial and Applied Mathematics.

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Allen S. Lange, for a photograph and biography, please see page 71 of the May, 1959, issue of these Transactions.

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Gordon J. Murphy (M'55) was born in Milwaukee, Wis., on February 16, 1927. He received the B.S. degree in electrical engineering from the Milwaukee School of Engineering, Milwaukee, Wis., in 1949. After having done graduate work in an evening program at the University of Wisconsin, Milwaukee, from 1949–1952, he received the M.S. degree in electrical engineering in 1952. He received the Ph.D. degree, also in electrical engineering, from the University of Minnesota, Minneapolis, in 1956.

G. J. Murphy

He was an assistant professor of electrical engineering at the Milwaukee School of Engineering until 1951, when he accepted a position as project engineer on inertial guidance systems with the AC Spark Plug Division of General Motors in Milwaukee. In 1952, he accepted a position as instructor in electrical engineering at the University of Minnesota. There he developed and taught courses in automatic control while continuing his graduate studies as a part-time student. In 1956 he was appointed assistant professor of electrical engineering at the University of Minnesota, and he held this position until 1957, when he became associate professor of electrical engineering in the Technological Institute of Northwestern University, Evanston, Ill. He has been engaged since then in teaching and in research in the fields of statistical control theory, sampled-data theory, and adaptive control.

Dr. Murphy is the author of two textbooks on automatic control.

He is a member of Sigma Xi, Eta Kappa Nu, AIEE, and ASEE.

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Joseph Otterman (S'52–M'56) was born in Warsaw, Poland, on April 12, 1925. He attended the Hebrew Institute of Technology (Technion), Haifa, Israel, and graduated in 1947 with the degree of Ingenieur in electrical engineering.

J. Otterman

From 1948 to 1950, he served in the Israeli Army, ending his military service as a lieutenant in the capacity of Engineering Reconnaissance Officer of the Command of the South. From 1950 to 1951, he worked in the Israeli Government Scientific Institute on developing instrumentation systems for ballistic measurements.

He came to the University of Michigan, Ann Arbor in 1951, and obtained the M.S.E. and Ph.D. degrees in electrical engineering from the University in in 1952 and 1955, respectively.

From 1955 to 1959, he was associated with the Engineering Research Institute, University of Michigan, working on rocket investigation of the upper atmosphere within the International Geophysical Year program, on navigation problems, and on problems of simulation by analog and digital techniques. In 1958 he proposed an experiment for determining upper atmosphere density and winds, utilizing a falling sphere and *xyz* accelerometers, and subsequently was engaged in investigation of this experiment under the sponsorship of the Air Force Cambridge Research Center, Bedford, Mass. He is currently with International Telephone and Telegraph Labs., Nutley, N. J., working at Bell Telephone Labs., Whippany, N. J., on topology and monitoring of communication systems.

Dr. Otterman is a member of the American Physical Society, Association for Computing Machinery, and Sigma Xi.

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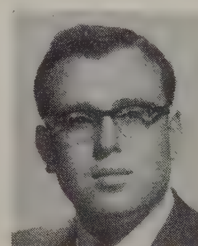
Hanoch Ur was born in Bucharest, Romania on October 16, 1931. He received the B.S. and Dipl.Ing. degrees from the Technion-Israel Institute of Technology, Haifa, in 1954 and 1955, respectively. From 1955 to 1956, he was a research assistant at the Computing Laboratory of Purdue University, La-Fayette, Ind., and received the M.S.E.E. degree from Purdue in 1956. After a short period as a logical designer with Univac, St. Paul, Minn., he joined the Polytechnic Institute of Brooklyn, Brooklyn, N. Y., as an instructor and graduate student. Recently he joined RCA in Camden, N. J., where he is working on high-speed computers.



H. Ur

Mr. Ur is a member of Sigma Xi.

PGAC News

Moscow Conference

Preparations for the first Congress of the International Federation of Automatic Control (IFAC) to be held in Moscow next June are being completed. In September, 1959, there was an IFAC General Assembly Meeting at Chicago, and Professor A. M. Letov, President-Elect of IFAC and Vice-Chairman of the U.S.S.R. National Committee of Automatic Control, gave his report on the "Preparation for the First IFAC Congress." With the approval of William E. Vannah, editor of *Control Engineering* and publicity chairman of the Congress, parts of Professor Letov's report are printed here to illustrate the progress that is being made and the difficulties that are encountered in the preparation of an international technical meeting.

I. Introduction

1) The First IFAC Congress is to be held in Moscow, June 27 to July 8, 1960, in accordance with the decision of the IFAC General Assembly held in Paris in September, 1957.

2) The scientific program of the Congress was worked out by the U.S.S.R. National Committee of Automatic Control. This program was approved by the IFAC Executive Council in Zurich in March, 1958. An outline of the program was published in the *IFAC Bulletin*, as well as in a special invitation distributed by Dr. Ruppel among IFAC national member organizations.

3) The U.S.S.R. National Committee continued the preparation for the Congress in conformity with this scientific program. The U.S.S.R. National Committee worked out a preliminary schedule of preparation for the Congress, of which the Executive Council was informed at the meeting in Rome in March, 1959.

4) This report outlines the picture of the future Congress, as well as proposes solutions of some organizational problems.

II. Selection of Papers

In accordance with the decision of the IFAC Executive Council passed in Rome in March, 1959, the U.S.S.R. National Committee, acting as the IFAC Papers Committee, in the spring of 1959 began its work of selecting papers for the Congress.

The U.S.S.R. National Committee has been happy to see very high activity displayed in Congress preparation by the majority of the IFAC member organizations. Four hundred and nineteen abstracts of papers have been received from nineteen countries.

The First IFAC Congress must demonstrate in the best way possible the scientific achievements of scientists and engineers that the Federation unites. Therefore the problem of selecting papers is at present one of the most responsible and most important problems. The difficulties in solving this problem lie in the fact that the U.S.S.R. National Committee has received more than twice the number of abstracts of papers that can be presented and discussed at the Congress technical sessions.

When considering these abstracts, the U.S.S.R. National Committee came to the conclusion that almost all of them touch upon very interesting problems of science and engineering, the elucidation of which would be most desirable.

By June 27, 1959, approximately 220 invitations had been sent to authors to submit the full texts of their papers to the U.S.S.R. National Committee. After June 27, 1959, the U.S.S.R. National Committee received 172 more abstracts. These abstracts were considered at the Committee's meeting on September 1, when a decision to send additional invitations was passed. A total of 321 invitations have now been sent.

But the selection of papers is not completed at this stage. The fact is that each abstract may be a good basis for an author to write a very interesting paper, and we expect all the authors invited to make use of this opportunity. However, there may be some cases in which the full paper proves to be of insignificant scientific interest, which will give reason for its rejection. For example, Soviet papers will be very strictly selected. We expect to select 60 to 70 full papers. We also passed a decision that each author be allowed to present only one paper at the Congress, just as the U.S.A. Automatic Control Committee has decided.

If we take into account that 10 to 15 per cent of the authors cannot attend the Congress for some reason or other, we shall have between 210 and 215 papers to be presented. These papers have been roughly classified into groups according to the problems treated, which enabled us to draft a program for the Congress sections.

Section I—Theory (7 groups of 141 papers),
Section II—Components (6 groups of 71 papers),
Section III—Application (6 groups of 118 papers).

Of the eleven days (the duration of the Congress), two days will be devoted to plenary sessions, three full days to visiting plants, and six days to presentation of papers and discussions. At present the U.S.S.R. National Committee is working out a detailed program of the Congress.

III. Nomination of Section Chairmen

After the final selection of papers to be presented has been completed in October, 1959, it will be advisable to appoint chairmen and vice-chairmen of the three sections from among the Congress participants, in conformance with the division of papers into groups.

Outstanding specialists in the fields covering scientific problems which refer to these groups must be invited to serve as section chairmen and vice-chairmen. The U.S.S.R. National Committee shall nominate possible candidates to serve as chairmen and vice-chairmen, work out a draft classification of scientific problems, and submit this information to the IFAC Executive Council for approval.

IV. Preprints, Publication, and Distribution

The U.S.S.R. National Committee has organized seven scientific-technical committees:

Theory—Prof. J. Z. Tsypkin;
Components—Prof. B. S. Sotskov;
Automatic Electric Drives—Prof. M. G. Chilikin;
Automation of Continuous Processes—E. P. Stefani;
Automation of Technological Processes—Academician V. I. Dikuskin;
Automation of Agricultural Production Processes—I. A. Budzko;
Terminology—Professor M. A. Gavrilov

to select abstracts of papers to be presented at the Congress, and they will be responsible for the preparation of the papers selected for publication; namely, translation of foreign papers into Russian, scientific editing, etc.

V. Organization of Simultaneous Translation

During the plenary sessions and the section sessions, the U.S.S.R. National Committee is arranging to have simultaneous translation made from English, French, and German into Russian, and from Russian into English only. It is very complicated to have simultaneous translation made from French and German into English. Therefore, the U.S.S.R. National Committee relies on the friendly assistance of Congress participants, who can speak various foreign languages, to organize simultaneous translation from French and German into English, and from Russian and English into French and German. It may happen that during the section sessions, the number of which exceeds three, some groups of the Congress participants will have no simultaneous translation available. The U.S.S.R. National Committee asks all national member organizations to inform the Congress Secretariat (Moscow I-53, Kalanchevskaja ul. 15 A) of all participants wishing to assist in a friendly way in organizing simultaneous translation at the Congress sessions.

VI. Congress Composition

The U.S.S.R. National Committee and the "Intourist" Agency have received a number of letters whose authors request to be allowed to attend the First IFAC Congress. Taking this into account, the U.S.S.R. National Committee draws the attention of the IFAC General Assembly, Executive Council, and member organizations to the necessity of very carefully forming the delegations to the Congress. The work of the Congress must not be interfered with a great number of curious tourists. Therefore, we

urge that each delegation consist of Congress speakers, specialists engaged in research work or teaching, and others whom the respective national organizations may confirm.

The national member organizations should now begin to form their delegations to the Congress, and then send the U.S.S.R. National Committee lists of delegates indicating the name and surname, place of work, permanent address. This information will be required for the U.S.S.R. National Committee to provide the delegates with hotel accommodations and other facilities. The information should reach the U.S.S.R. National Committee not later than February, 1960.

Secretariat, Press Department, Excursion Office, Chief of the Congress Program

The U.S.S.R. National Committee is organizing the Congress Secretariat, the Press Department, the Excursion Office, and appointing the chief of the Congress Program.

Exhibition of Scientific and Technical Literature

In accordance with inquiries received from various people, institutions and firms, the U.S.S.R. National Committee agrees in principle with the proposal to organize an exhibition of scientific and technical literature during the Congress.

VII. Congress Registration

In order to reimburse partially the expenditure the U.S.S.R. National Committee is to bear, and in conformity with international traditions, it is necessary to establish Congress membership dues of $20 before March 31 and $30 afterward for people who are not presenting papers. Authors may register for $15 before that date, and for $20 later.

Registration paid in advance of March 31 will entitle registrants to a full set of preprints in English. Butterworths Scientific Publications, the *Proceedings* publisher, will distribute the preprints some time in April.

A. M. Letov
President-Elec of IFAC
Vice-Chairman of U.S.S.R. National Committee of Automatic Control

If You Are Going to IFAC'S Congress . . .

For information on arrangements for attendance at the First IFAC Congress, Moscow, U.S.S.R., June 27–July 8, 1960, write to:

Secretary, American Automatic Control Council
(Editor, *Control Engineering*)
330 West 42nd Street
New York 36, N. Y.

Inform him of your plans to attend the Congress by the first week in February. AACC needs your full name, title, place of work, and address so that it may assemble for IFAC a complete list of people who will attend from the U.S.A.

1960 National Control Conference

The 1959 National Control Conference was sponsored by the IRE with official participation by the AIEE, ISA, and ASME. The 1960 National Control Conference will be held in Boston, Mass., during September, and it will be sponsored by the ASME with participation of the IRE, AIEE, and the ISA. Harvey A. Miller, Manager, Electronic Development Division, Taylor Instrument Companies, 95 Ames Street, Rochester 1, N. Y., will accept papers for the IRE-PGAC members who wish to participate in the Conference. Further details about the Conference can be obtained from Mr. Miller.

INSTITUTIONAL LISTINGS

The IRE Professional Group on Automatic Control is grateful for the assistance given by the firms listed below, and invites application for Institutional Listings from other firms interested in the field of Automatic Control.

The charge for an Institutional Listing is $75.00 per issue or $125.00 for two consecutive issues. Applications for Institutional Listings and checks (made out to the Institute of Radio Engineers, Inc.) should be sent to Mr. L. G. Cumming, Technical Secretary, Institute of Radio Engineers, Inc., 1 East 79th Street, New York 21, N. Y.